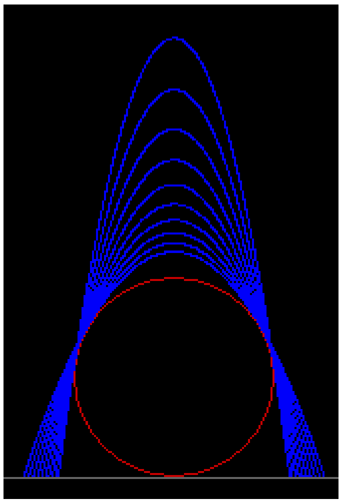
Natural Log

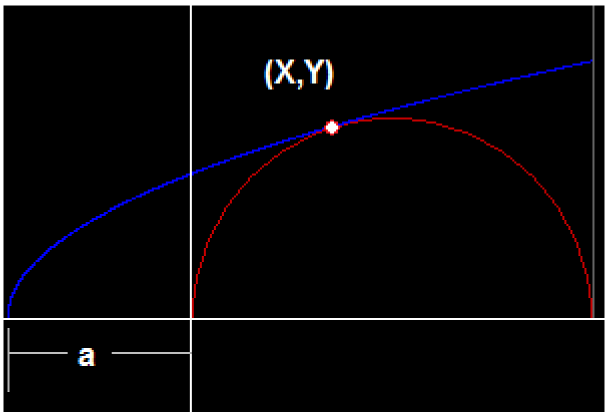
Problem: Published as a challenge to students and teachers in the November 2010 issue of *The Physics Teacher*.

The idea is that a flea jumps over a circular obstacle (think a log) and the question is what the most efficient angle is, so that it requires the slowest initial speed.

Solution: Some possible trajectories of the flea are shown in the figure below. To minimize the energy needed the trajectories are tangent to the log at two points. It is also possible to find a trajectory that is only tangent at the top, but as we will see below that is not the best trajectory.



As the problem is symmetric, it can be analyzed on one half of the log; moreover I would prefer to rotate the trajectory 90 degrees as shown here:



Given "a" and the radius "R" there is only one parabola that will be tangent to the circle. To find it we notice that at the common point (x, y), where the functions intersect, their derivatives are also identical.

The functions are

Circle: $y=\sqrt{R^2-(R-x)^2}$

Parabola: $y=b\sqrt{x+a}$

But the parabola can also be written from the point of view of free fall as follows:

$$y=v_o\sqrt{\frac{2(x+a)}{g}}=v_o\sqrt{\frac{2}{g}}\sqrt{x+a}$$

The two conditions for a common point (X,Y) with equal derivatives are:

$$b\sqrt{X+a}=\sqrt{R^2-(R-X)^2} \quad \text{And} \quad \frac{b}{2\sqrt{X+a}}=\frac{R-X}{\sqrt{2RX-X^2}}$$

Multiplying these two equations: $\frac{b^2}{2}=R-X \rightarrow X=R-\frac{b^2}{2}$

And dividing the equations:

$$2(X+a)=\frac{2RX-X^2}{R-X} \rightarrow 2\left(RX-X^2+aR-aX\right)=2RX-X^2 \rightarrow -X^2+2aR-2aX=0$$

Replacing X in this last equation: $-\left(R-\frac{b^2}{2}\right)^2+2aR-2a\left(R-\frac{b^2}{2}\right)=0$

Simplifying: $a=\frac{\left(R-\frac{b^2}{2}\right)^2}{b^2}$, but $b=v_o\sqrt{\frac{2}{g}} \rightarrow \frac{b^2}{2}=\frac{v_o^{\,2}}{g}$ So: $a=\frac{\left(R-\frac{v_o^{\,2}}{g}\right)^2}{2\frac{v_o^{\,2}}{g}}$

Now, the total energy is: $T.E.=\frac{1}{2}mv_o^{\,2}+mg(2R+a)$

So we need to minimize $f=\frac{v_o^{\,2}}{g}+2a$ to get the minimum energy.

$$f = \frac{v_o^{\ 2}}{g} + \frac{R^2 - 2R\frac{v_o^{\ 2}}{g} + \frac{v_o^{\ 4}}{g^2}}{\frac{v_o^{\ 2}}{g}} = 2\frac{v_o^{\ 2}}{g} - 2R + \frac{R^2 g}{v_o^{\ 2}}$$

Taking derivative: $f' = 4\frac{v_o}{g} - \frac{2R^2 g}{v_o^{\ 3}} = 0 \rightarrow \boxed{v_o = \sqrt{\frac{Rg}{\sqrt{2}}}}$

The value of "a" can be calculated from this: $a = \left(\frac{3\sqrt{2}-4}{4}\right)R$

The initial velocity of the projectile will be:

$$v_1 = \sqrt{v_o^{\ 2} + 2ag + 4Rg} = \sqrt{\frac{Rg}{\sqrt{2}} + 2\left(\frac{3\sqrt{2}}{4} - 1\right)Rg + 4Rg} = \sqrt{Rg}\sqrt{2\sqrt{2}+2}$$

The initial angle: $\theta_o = \cos^{-1}\left(\frac{v_o}{v_1}\right) = \cos^{-1}\left(\frac{1}{\sqrt{4+2\sqrt{2}}}\right) = 67.5^\circ$

The distance from the point of contact of the log with the ground to the initial position of the flea:

$$L = \frac{v_1^{\ 2}}{g}\sin\theta\cos\theta = \frac{\sqrt{6+4\sqrt{2}}}{2}R = \frac{\sqrt{4+2\sqrt{8}+2}}{2}R = \left(1+\frac{\sqrt{2}}{2}\right)R$$