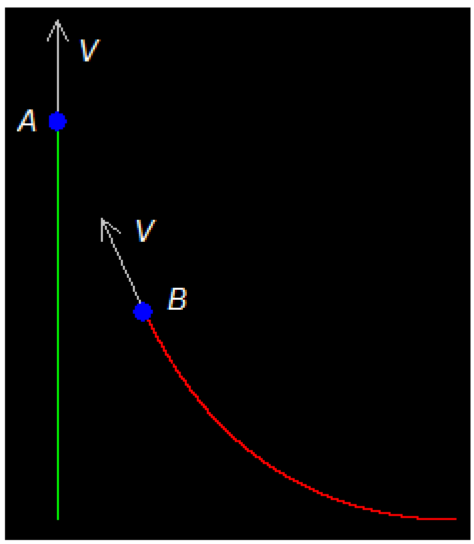
A futile chase

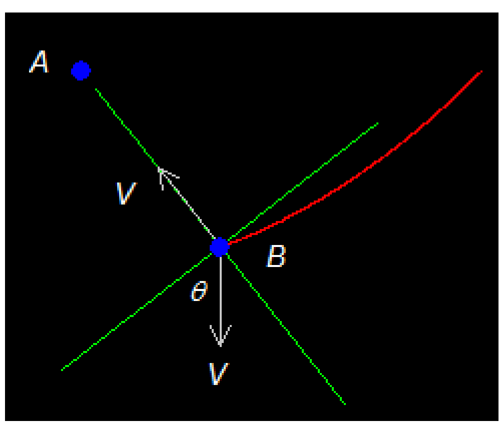
Problem: Published as a challenge to students and teachers in the September 2012 issue of *The Physics Teacher*.

There are two turtles initially separated by a distance d. Turtle A moves in a straight line with constant velocity V and turtle B always follows turtle A with constant speed V. The question is how much will be the separation between the turtles after a long time. The initial conditions in the magazine had A's velocity perpendicular to the vector that separates the turtles, but we can generalize this.

Solution: As the chase progresses turtle A's velocity will always be in the positive y-direction while turtle B's velocity will always be directed towards A.



We change the origin of coordinates to the position of turtle A. Then the relative velocity of turtle B will always be the sum of two vectors: V in the negative y-direction and V in the radial direction towards the origin as shown in the figure:



The relative velocity can be written in terms of radial and tangential components using angle θ shown in the figure as variable.

$$v_r = \frac{dr}{dt} = -v + v\sin\theta$$
$$v_\theta = r\frac{d\theta}{dt} = v\cos\theta$$

To solve the equations we can divide one by the other, which leaves a differential equation that can be separated:

$$\frac{v_r}{v_\theta} = \frac{\frac{dr}{dt}}{r\frac{d\theta}{dt}} = \frac{-v + v\sin\theta}{v\cos\theta} \to \frac{dr}{rd\theta} = \frac{-1+\sin\theta}{\cos\theta} \to \frac{dr}{r} = \frac{-1+\sin\theta}{\cos\theta}d\theta$$

We can integrate the two sides of the equation with the understanding that *r* will start at *d* and finish at a final value d_f and the angle will start at zero and finish at π/2. Notice that when the angle is π/2 the relative velocity will be zero, so the second turtle will approach that angle after a long time.

The integrals are:

$$\int_d^{d_f} \frac{dr}{r} = \int_0^{\pi/2} \frac{-1+\sin\theta}{\cos\theta}d\theta \to \ln\frac{d_f}{d} = 2\ln\frac{\sqrt{2}}{2} \to d_f = \frac{d}{2}$$

So, the solution is that after a long time the distance between the turtles will be d/2.

A generalization: The solution above offers us a more general case: If turtle A’s velocity is not in the positive y-direction, but at an angle φ with the positive x-axis, then the distance after a long time will be:

$$d_f = d\sin^2(\phi/2)$$