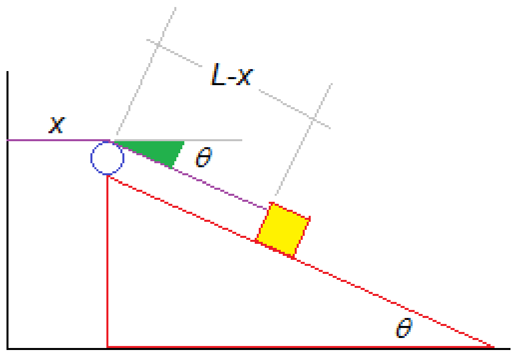
Another wedge issue

Problem: Published as a challenge to students and teachers in the October 2012 issue of *The Physics Teacher*.

There is a wedge of mass m on a smooth surface and a block also of mass m on the wedge connected to a string as in the figure below. The question is the acceleration of the wedge as the block slides down without friction.



Solution: We define the left end of the string as the origin of coordinates, define variable x as the position of the top of the pulley. We also define L as the length of the string.

Vector (x,0) can be used to specify the position of the wedge, so:

$$x_w = x$$
$$y_w = 0$$

For the block the position can be specified as follows

$$x_b = x + (L - x)\cos\theta$$
$$y_b = -(L - x)\sin\theta$$

We take derivatives with respect to time to find velocities

$$\dot{x}_w = \dot{x}$$
$$\dot{y}_w = 0$$

$$\dot{x}_b = \dot{x}(1 - \cos\theta)$$
$$\dot{y}_b = \dot{x}\sin\theta$$

Then we calculate the kinetic energy of the system

$$K.E. = \frac{1}{2}m(\dot{x}_b{}^2 + \dot{y}_b{}^2) + \frac{1}{2}m(\dot{x}_w{}^2 + \dot{y}_w{}^2) = \frac{1}{2}m\dot{x}^2(1 + (1 - \cos\theta)^2 + \sin^2\theta)$$

This can be simplified $K.E. = \frac{1}{2} m\dot{x}^2 (3 - 2\cos\theta)$

The potential energy is $P.E. = -mg(L - x)\sin\theta$, where the zero is taken when *x*=*L*

Now, the change in kinetic energy comes from the change in potential energy, so:

$$mg(L - x)\sin\theta = \frac{1}{2} m\dot{x}^2 (3 - 2\cos\theta)$$

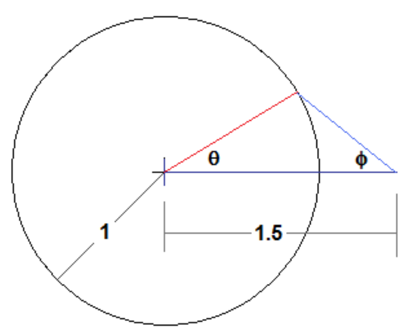
this equation is now simplified and differentiated with respect to time:

$$-g\dot{x}\sin\theta = \ddot{x}\dot{x}(3 - 2\cos\theta) \rightarrow \boxed{\ddot{x} = -\frac{g\sin\theta}{3 - 2\cos\theta}}$$

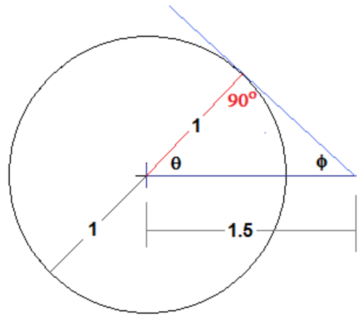
Here the minus sign indicates acceleration to the left.

Extension to the problem: Now that we have the acceleration of the wedge it'd be interesting to see what angle produces the maximum acceleration (it isn't 90 degrees!):

We could take the derivative with respect to θ and make it zero, but a geometrical construction will answer that question too. To do this we rewrite the acceleration as $\ddot{x} = -\frac{g}{2}\left[\frac{\sin\theta}{1.5 - \cos\theta}\right]$ where the quantity in brackets is the tangent of ϕ in the following diagram.



To maximize the value we need to make the right side of the triangle tangent to the unit circle as follows



So the maximum acceleration will happen when $\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$

And the maximum acceleration is $\ddot{x} = -\frac{g}{2}\tan\phi = -\frac{g}{2}\left[\frac{1}{\sqrt{1.5^2 - 1}}\right] = -\frac{\sqrt{5}}{5}g$