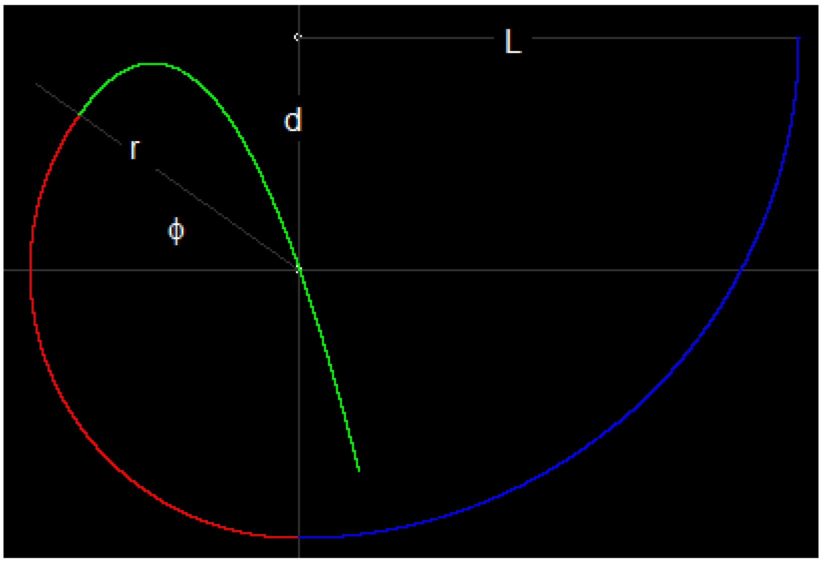
The Pin and the Pendulum

Problem: Published as a challenge to students and teachers in the November 2012 issue of *The Physics Teacher*.

A pendulum of length L is pulled to the side and released. A pin located at a distance d directly below the center of rotation of the pendulum restricts the motion to a smaller circle of radius L-d. We want to find the minimum value of d, so the pendulum wraps around the pin at least once.



We divide the trajectory in three parts.

In the first part, shown in blue in the figure, the trajectory is one fourth of a circle and due to conservation of mechanical energy the velocity at the end of the trajectory is $v = \sqrt{2gL}$

In the second part, shown in red in the figure, the pin restricts the radius to $r = L - d$ and the trajectory will remain circular until the string becomes slack.

The condition of the string changing from taut to slack can be written as

$$m\frac{v^2}{r} - mg\sin\phi = 0$$

Here the first term is the centripetal force necessary to maintain a circular trajectory and the second term is the component of the weight in the direction of the string. If the sum of these forces is zero the force on the string will be zero. With this equation we can find the angle ϕ

$$\sin\phi = \frac{v^2}{rg} \ldots (1)$$

Due to conservation of energy we also know that

$$\frac{1}{2}mv^2 + mg(r + r\sin\phi) = \frac{1}{2}m(\sqrt{2gL})^2$$

So, $\frac{1}{2}v^2 + g(r + r\sin\phi) = gL \ldots(2)$

Replacing (1) in (2) we get: $\frac{1}{2}v^2 + g(r + \frac{v^2}{g}) = gL \rightarrow v = \sqrt{\frac{2gd}{3}}$

This last equation allows us to find the angle ϕ at the end of the second part of the trajectory:

$$\sin\phi = \frac{v^2}{rg} = \frac{2d}{3r}$$

and using the pin as origin of coordinates these are the conditions at the end of the second part of the trajectory:

$$x = -r\cos\phi = -\frac{\sqrt{9r^2 - 4d^2}}{3}$$

$$y = r\sin\phi = \frac{2d}{3}$$

$$v_x = \sqrt{\frac{2gd}{3}}\frac{2d}{3r}$$

$$v_y = \sqrt{\frac{2gd}{3}}\sqrt{1 - \frac{4d^2}{9r^2}}$$

In the third part, shown in green in the figure, the trajectory will be parabolic as it is free fall. The time to reach the vertical axis will be:

$$t = \frac{x}{v_x} = \frac{\frac{\sqrt{9r^2 - 4d^2}}{3}}{\sqrt{\frac{2gd}{3}}\frac{2d}{3r}} = \frac{r\sqrt{3(9r^2 - 4d^2)}}{2d\sqrt{2gd}}$$

And at that time *y* will be:

$$y=\frac{2d}{3}+\sqrt{\frac{2gd}{3}}\sqrt{1-\frac{4d^2}{9r^2}}\frac{r\sqrt{3(9r^2-4d^2)}}{2d\sqrt{2gd}}-\frac{1}{2}g\left[\frac{r\sqrt{3(9r^2-4d^2)}}{2d\sqrt{2gd}}\right]^2$$

The condition to wrap around the pin is that $y>0$, so

$$\frac{2d}{3}+\sqrt{\frac{2gd}{3}}\sqrt{1-\frac{4d^2}{9r^2}}\frac{r\sqrt{3(9r^2-4d^2)}}{2d\sqrt{2gd}}-\frac{1}{2}g\left[\frac{r\sqrt{3(9r^2-4d^2)}}{2d\sqrt{2gd}}\right]^2>0$$

After algebraic simplifications we get $4d^2-3r^2>0$, so the condition becomes

$$d>\frac{\sqrt{3}r}{2}=\frac{\sqrt{3}}{2}(L-d)\rightarrow\boxed{d>(2\sqrt{3}-3)L}$$