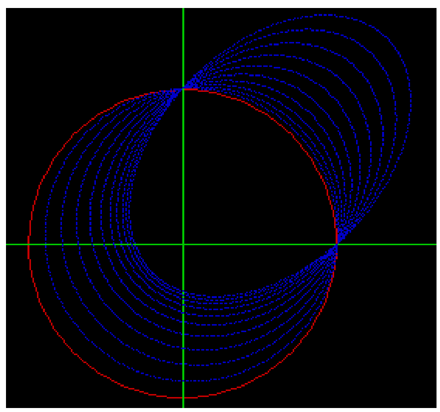
One Giant Leap

Problem: Published as a challenge to students and teachers in the December 2012 issue of *The Physics Teacher*.

A projectile is launched from the north pole of a spherical planet with the minimum initial speed to reach the equator. The question is what that initial speed is taking into account that the escape velocity is v_{escape}.

Solution: After the launch with initial speed v_i the trajectory of the projectile will be an ellipse with one focal point at the center of the planet. The other focal point has to be at an angle of 45 degrees. We know this because the landing point is at 90 degrees and that point is also part of the ellipse and symmetric. In other words, all possible trajectories are ellipses with the major axis at 45 degrees off the axis of rotation. Some ellipses are shown in the figure below.

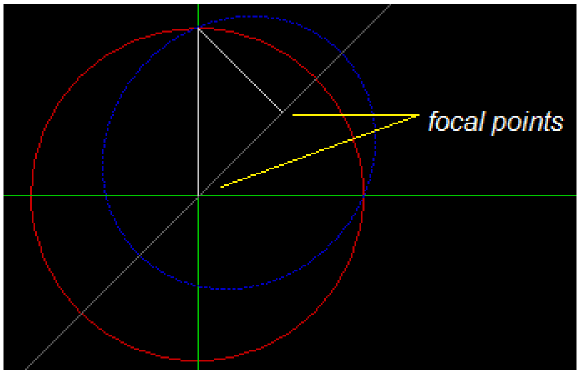


The total energy of a projectile in an elliptical trajectory only depends on the length of the major axis $E = -\frac{GMm}{2a} = \frac{1}{2}mv_i^2 - \frac{GMm}{R}$. That means that to minimize the speed at launch we should minimize a, because $v_i = \sqrt{\frac{2GM}{R}}\sqrt{1-\frac{R}{2a}} = v_{escape}\sqrt{1-\frac{R}{2a}}$

Graphically we know that $2a$ is the sum of the distances from any point in the ellipse to the foci. One distance is already fixed at R, but the other distance is variable. The shortest value is $R/\sqrt{2}$ as shown in the next figure.

This best case corresponds to $a = R\left(\frac{2+\sqrt{2}}{4}\right)$, which substituted in the energy equation gives

$$\boxed{v_i = v_{escape}\sqrt{\sqrt{2}-1}}$$



Also interestingly, the angle of elevation of the projectile at launch is 22.5 degrees. You can deduce this if you notice that any tangent to the ellipse makes the same angle with the lines that connect the tangent point to the two foci.

Extension to the problem

What if the point that we want to hit with the projectile is not on the equator? Let's say that it is at latitude λ with a positive number indicating north. Then the major axis of the ellipse would be at an angle of $45^\circ - \lambda/2$ with respect to the axis of rotation, the shortest value of a would be $a = R\left[\dfrac{1+\sin(45^\circ - \lambda/2)}{2}\right]$ with the minimum speed

$$v_i = \frac{v_{escape}}{\sqrt{1+\sec(45^\circ + \lambda/2)}}$$

And the angle of elevation for this trajectory is $\phi = 22.5^\circ + \lambda/4$

In the extreme case where we wanted to reach the south pole, the value of the minimum speed would be $v_i = \dfrac{v_{escape}}{\sqrt{2}}$, which corresponds to a circular trajectory as in a satellite.

The other extreme case of a latitude close to 90 degrees north can be analyzed as follows: Let's change variables to $\alpha = 90 - \lambda$, so then $v_i = \dfrac{v_{escape}}{\sqrt{1+1/\sin(\alpha/2)}} \approx v_{escape}\sqrt{\sin(\alpha/2)} \approx v_{escape}\sqrt{\alpha/2}$, where the angle is in radians. If the distance between the north pole and the target is d, then the initial speed will be $v_i = \sqrt{\dfrac{2GM}{R}}\sqrt{\dfrac{d}{2R}} = \sqrt{\dfrac{GMd}{R^2}} = \sqrt{gd}$, which is the same result that you get with the elementary equation $d = \dfrac{v_i^{\,2}}{g}\sin 2\phi$ using 45 degrees for the angle of elevation ϕ

Two examples:

Target at latitude 30 degrees south and 30 degrees north