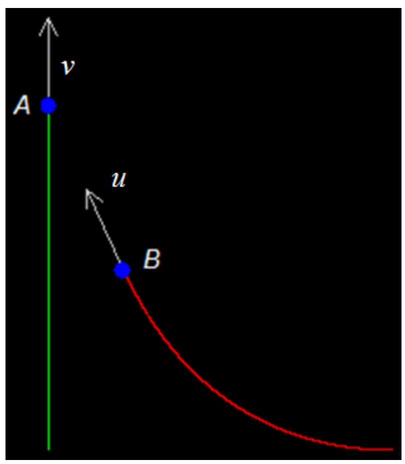
Fox Trot

Problem: Published as a challenge to students and teachers in the September 2017 issue of *The Physics Teacher*.

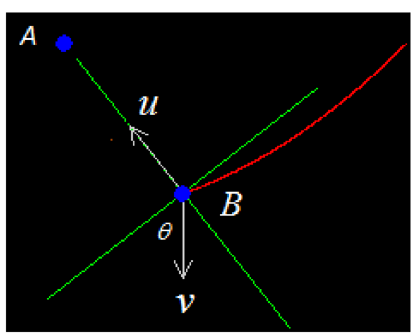
You have a rabbit chasing a fox (yes, not the other way around). The fox moves in a straight line with constant velocity v and the rabbit has a constant speed u, but his velocity is always directed towards the fox. Initially the distance between them is L and move with velocities perpendicular to each other.

We want to know the distance covered by the rabbit to reach the fox.

Solution: We take a coordinate system where the fox moves in the y-axis and the rabbit is initially on the x-axis at a distance L from the origin. As the chase progresses fox A's velocity will always be in the positive y-direction while rabbit B´s velocity will always be directed towards A.



We change the origin of coordinates to the position of A. Then the relative velocity of B is the sum of two vectors: v in the negative y-direction and u in the radial direction towards the origin as shown in the figure:



The relative velocity can be written in terms of radial and tangential components using angle θ shown in the figure as variable.

$$v_r = \frac{dr}{dt} = -u + v\sin\theta$$
$$v_\theta = r\frac{d\theta}{dt} = v\cos\theta \quad \ldots.(*)$$

To solve the equations we can divide one by the other, which leaves a differential equation that can be separated:

$$\frac{v_r}{v_\theta} = \frac{\frac{dr}{dt}}{r\frac{d\theta}{dt}} = \frac{-u + v\sin\theta}{v\cos\theta} \rightarrow \frac{dr}{rd\theta} = \frac{-k+\sin\theta}{\cos\theta} \rightarrow \frac{dr}{r} = \frac{-k+\sin\theta}{\cos\theta}d\theta$$

Here k is the ratio *u/v*. Integrating:

$$\int_L^r \frac{dr}{r} = \int_0^\theta \frac{-k+\sin\theta}{\cos\theta}d\theta$$

$$r = \frac{L}{\cos\theta}\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^{k/2}$$

We get *r=0* when θ reaches π/2, which we will use as the limit of integration below.

Replacing *r* in equation (*)

$$\frac{L}{v\cos\theta\cos\theta}\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^{k/2} d\theta = dt$$

Integrating

$$t = \frac{L}{v}\int_0^{\pi/2} \frac{1}{\cos\theta\cos\theta}\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^{k/2} d\theta$$

Change of variables to integrate

$$t = \frac{L}{2v}\int_0^{\pi/4} \tan^{k-2}\theta(1+\tan^2\theta)d\tan\theta$$

$$t = \frac{L}{2v}\int_0^1 x^{k-2}(1+x^2)dx$$

$$t = \frac{L}{2v}\left(\frac{1}{k-1} + \frac{1}{k+1}\right) = L\left(\frac{u}{u^2 - v^2}\right)$$

Finally we multiply the time by the speed to get the distance:

$$d = L\left(\frac{u^2}{u^2 - v^2}\right)$$