Just pulling through…

Problem: Published as a challenge to students and teachers in the October 2017 issue of *The Physics Teacher*.

A marble of mass m can slide without friction along a horizontal rod.

The marble is pulled by a light string of length L so that the velocity of the "loose" end of the string is always directed along the string and has a constant magnitude v.

What is the magnitude of the force applied to the string when it makes angle θ with the rod?

Solution: To solve the problem we consider that the marble has coordinates (X,0)

And the loose end of the string has the coordinates (x,y) given by the equations

$$x = X + L\cos\theta$$
$$y = L\sin\theta$$

Taking derivatives to these coordinates and equating them to the components of the velocity we get these equations

$$\frac{dx}{dt} = \frac{dX}{dt} - \frac{d\theta}{dt} L\sin\theta = v\cos\theta$$
$$\frac{dy}{dt} = \frac{d\theta}{dt} L\cos\theta = v\sin\theta \qquad \text{… equations (*)}$$

We notice that the velocity of the marble has this simple form:

$$\frac{dX}{dt} = v\cos\theta + v\sin\theta\tan\theta = v\sec\theta$$

Taking another derivative we get the acceleration of the marble.

$$\frac{d^2X}{dt^2} = v\frac{\sin\theta}{\cos^2\theta}\frac{d\theta}{dt}$$

Here we can substitute the second equation (*) getting

$$\frac{d^2X}{dt^2} = \frac{v^2}{L}\frac{\sin^2\theta}{\cos^3\theta}$$

To get the net force on the mable we multiply its acceleration by its mass

$$F_{net} = m\frac{d^2X}{dt^2}$$

But this net force is the horizontal component of the force applied by the string, so the force applied by the string is

$$F = \frac{m}{\cos\theta}\frac{d^2X}{dt^2}$$

And substituting our earlier result for the acceleration we get

$$F = \frac{mv^2}{L}\frac{\sin^2\theta}{\cos^4\theta}$$

It is interesting to notice that if the initial angle is not zero, this situation cannot be maintained forever…