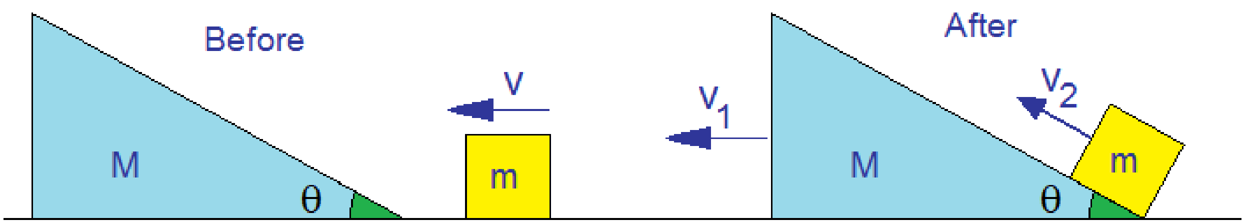
Plane and Simple

Problem: Published as a challenge to students and teachers in the December 2018 issue of *The Physics Teacher*.

Initially a small block of mass m is moving with speed v towards a larger block of mass M shaped like a wedge. The surfaces are all frictionless and the small block slides up and down the slope. The transition to the inclined plane is smooth. Calculate the time of the small block on the plane if M=4m.

Solution:

First we consider the smooth transition of the m-block to the inclined plane.



During this transition, kinetic energy and momentum in the horizontal direction are conserved, so we can write:

$$mv = (M+m)v_1 + mv_2\cos\theta \qquad (1)$$

$$\frac{1}{2}mv^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}m(v_1^2 + v_2^2 + 2v_1v_2\cos\theta) \qquad (2)$$

Where v_1 is the velocity of the inclined plane and v_2 is the relative velocity of the m-block with respect to the inclined plane.

These equations can be solved to get v_1 and v_2:

$$v_1 = \frac{mv}{M+m}\left(1 - \sqrt{\frac{M}{M+m\sin^2\theta}}\cos\theta\right) \qquad (3)$$

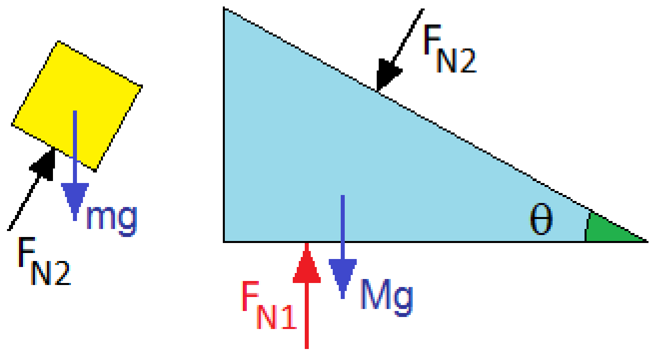
$$v_2 = v\sqrt{\frac{M}{M+m\sin^2\theta}} \qquad (4)$$

After the m-block climbs up the inclined plane and gets back, the final velocities will be

$$v_1 = \frac{mv}{M+m}\left(1+\sqrt{\frac{M}{M+m\sin^2\theta}}\cos\theta\right) \qquad (5)$$

$$v_2 = -v\sqrt{\frac{M}{M+m\sin^2\theta}} \qquad (6)$$

With the m-block on the inclined plane the forces will be:



Choosing a set of coordinate axes where left is X and up is Y, Newton´s second law gives us the equations:

$$F_{N2}\cos\theta - mg = ma_y \qquad (7)$$

$$-F_{N2}\sin\theta = ma_x \qquad (8)$$

$$F_{N2}\sin\theta = Ma'_x \qquad (9)$$

Where a´ is the acceleration of the M-block.

We can combine the last two equations to get:

$$a'_x = -\frac{m}{M}a_x \qquad (10)$$

We get one more equation from geometric considerations. The m-block´s relative velocity has to be parallel to the inclined plane, so:

$$v_y = (v_x - v'_x)\tan\theta \qquad (11)$$

Taking derivative with respect to time:

$$a_y = (a_x - a'_x)\tan\theta \qquad (12)$$

Replacing a'_x from equation 10 here, we get

$$a_y = a_x\left(1+\frac{m}{M}\right)\tan\theta \tag{13}$$

We now eliminate F_{N2} from equations (7) and (8) and replace equation (13) to get:

$$\tan\theta = \frac{-a_x}{a_x(1+m/M)\tan\theta + g} \tag{14}$$

And solving for a_x we get

$$a_x = -\frac{M\tan\theta}{M+(M+m)\tan^2\theta}\,g \tag{15}$$

So a'_x is

$$a'_x = \frac{m\tan\theta}{M+(M+m)\tan^2\theta}\,g \tag{16}$$

We can now calculate the time with the initial and final velocities of block M, equations 3 and 5, and the acceleration of equation 16.

$$t = \frac{2v}{g}\frac{\sqrt{M(M+m\sin^2\theta)}}{(M+m)\sin\theta} \tag{17}$$

For the particular case M=4m, the time is:

$$t = \frac{4v}{5g}\sqrt{1+4\csc^2\theta}$$