Putnam Problem with a cubic function

Problem B1 in the Putnam competition in 2006 went like this:

Show that the curve

$$x^3 + y^3 + 3xy = 1$$

contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

We notice that this curve is symmetric about the line x=y because if you replace x with y the equation remains the same. Then, if only one equilateral triangle exists with its three corners on the curve we can conclude that due to symmetry one corner has to be on the x=y line.

Solving the equation when x=y will give us the position of that corner:

$$x^3 + y^3 + 3xy = 1 \wedge x = y \rightarrow 2x^3 + 3x^2 - 1 = 0 \rightarrow (x+1)^2(2x-1) \rightarrow x = -1 \; or \; x = 1/2$$

If (-1,-1) is one corner of the triangle, then another corner has to be at

$$(-1+l, -1+l\tan 15^\circ)$$

Where, starting from point (-1,-1) we added a vector $(\ell, \ell\tan 15^\circ)$ to get to another corner of the triangle.

This point has to be on the curve, so it has to satisfy the equation:

$$(-1+l)^3 + (-1+l\tan 15^\circ)^3 + 3(-1+l)(-1+l\tan 15^\circ) = 1$$

And after simplifying we get

$$-1+3l-3l^2+l^3-1+3l\tan 15^\circ - 3l^2\tan^2 15^\circ + l^3\tan^3 15^\circ + 3 - 3l + 3l^2\tan 15^\circ - 3l\tan 15^\circ = 1$$

$$-3l^2 - 3l^2\tan^2 15^\circ + 3l^2\tan 15^\circ + l^3 + l^3\tan^3 15^\circ = 0 \rightarrow l = 0 \; or \; l = \frac{3}{1+\tan 15^\circ}$$

The solution $\ell = 0$ is a trivial solution because it only gives the same initial point (-1,-1) as the corner. The other solution: $l = \frac{3}{1+\tan 15^\circ}$, indicates an equilateral triangle with side

$$s = \frac{3}{(1+\tan 15^\circ)\cos 15^\circ},$$

which means that the area of the triangle is: $Area = \frac{\sqrt{3}s^2}{4} = \frac{9\sqrt{3}}{4(1+\tan 15^\circ)^2 \cos^2 15^\circ}$

The trigonometric functions of 15 degrees are known, so we can replace them in the formula above to give the close solution:

$$Area = \frac{9\sqrt{3}}{4\left[1+2-\sqrt{3}\right]^2 \frac{2+\sqrt{3}}{4}} = \frac{3\sqrt{3}}{2}$$