Putnam Problem and Richard Feynman

Problem A5 in the Putnam competition in 2005 was calculating the following integral

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$$

When I saw this problem I immediately thought of Richard Feynman and his book "*Surely you are joking…*" In one of the chapters he describes an integration technique by derivative under the integral sign and yes! It works in this case. Let's see how it is done:

Let's define a function of a new variable $f(y) = \int_0^1 \frac{\ln(yx+1)}{x^2+1}\,dx$

This will allow us to take a derivative. Notice that the original integral is this function evaluated at y=1, also notice that

$$f(0) = \int_0^1 \frac{\ln(1)}{x^2+1}\,dx = 0$$

Now, with the derivative of this function:

$f'(y) = \int_0^1 \frac{x}{x^2+1}\frac{1}{xy+1}\,dx$, which can be expanded in the following terms:

$$f'(y) = \frac{y}{y^2+1}\int_0^1 \frac{1}{x^2+1}\,dx + \frac{1}{y^2+1}\int_0^1 \frac{x}{x^2+1}\,dx - \frac{y}{y^2+1}\int_0^1 \frac{1}{xy+1}\,dx$$

All of the terms are easily integrated, so we get:

$$f'(y) = \frac{y}{y^2+1}\tan^{-1}(x) + \frac{1}{2}\frac{1}{y^2+1}\ln(x^2+1) - \frac{1}{y^2+1}\ln(xy+1)\Big|_{x=0}^{x=1}$$

$$f'(y) = \frac{y}{y^2+1}\frac{\pi}{4} + \frac{1}{2}\frac{1}{y^2+1}\ln(2) - \frac{1}{y^2+1}\ln(y+1)$$

Since $f(0) = 0$, to find $f(1) = I$ we need to integrate $f'(y)$ from zero to one

$$I = \int_0^1 \left[\frac{y}{y^2+1}\frac{\pi}{4} + \frac{1}{2}\frac{1}{y^2+1}\ln(2) - \frac{1}{y^2+1}\ln(y+1)\right]dy$$

We notice that the third term is equal to I by its own definition, so it can be taken to the left side of the equation and dividing by 2 we get:

$$I = \frac{1}{2}\int_0^1 \left[\frac{y}{y^2+1}\frac{\pi}{4} + \frac{1}{2}\frac{1}{y^2+1}\ln(2) \right] dy = \frac{1}{2}\left[\frac{\pi}{4}\frac{1}{2}\ln(y^2+1) + \frac{1}{2}\ln(2)\tan^{-1}(y) \right]_{y=0}^{y=1}$$

Finally: $I = \frac{\pi}{8}\ln(2)$