Putnam Problem with a torus

Problem A1 in the Putnam competition in 2006 went like this:

Find the volume of the region of points such that

$$\left(x^2 + y^2 + z^2 + 8\right)^2 \leq 36(x^2 + y^2)$$

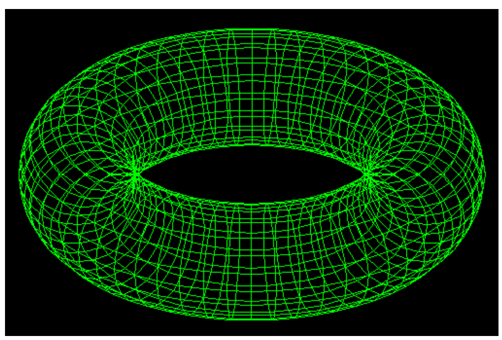
To simplify the expression we could use cylindrical coordinates and change variables $r = \sqrt{x^2 + y^2}$, so the expression becomes: $\left(r^2 + z^2 + 8\right)^2 \leq 36r^2$

Taking square root we get $r^2 + z^2 + 8 \leq 6r$, we are justified in doing this because both sides of the equation are positive.

Next we rearrange terms as follows:

$r^2 + z^2 + 8 \leq 6r \rightarrow r^2 - 6r + z^2 + 8 \leq 0 \rightarrow (r-3)^2 + z^2 \leq 1$

This last equation describes a circle in any r-z plane, so the volume is a torus. The cross section of the torus is a circle of radius 1 and the torus itself has a radius of 3.



The volume can be calculated with the use of Pappus' theorem: To calculate the volume of a solid of revolution you multiply the area of the cross section (in this case π) times the length of the circumference described by the center of mass of the cross section (in this case 6π), so the volume is $6\pi^2$