Calculus

Asymptotes

Problem 1.- Find the asymptotes of

$$f(x)=\frac{3x^2+5x+2}{\sqrt{16x^4-81}}$$

Solution: To find the horizontal asymptotes we examine the limits at $+\infty$ and $-\infty$:

$$\lim_{x\to\infty}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}=\lim_{x\to\infty}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}\times\frac{1/x^2}{1/x^2}$$

$$\lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{2}{x^2}}{\sqrt{16-\frac{81}{x^4}}}=\frac{3+0+0}{\sqrt{16-0}}=\frac{3}{4}$$

$$\lim_{x\to-\infty}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}=\lim_{x\to-\infty}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}\times\frac{1/x^2}{1/x^2}$$

$$\lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{2}{x^2}}{\sqrt{16-\frac{81}{x^4}}}=\frac{3+0+0}{\sqrt{16-0}}=\frac{3}{4}$$

Notice that in this case, $\frac{1}{x^2}=\frac{1}{\sqrt{x^4}}$, there is no change in sign because x^2 is positive.

The asymptotes are the same straight line y=3/4 for both limits $+\infty$ and $-\infty$

Now, the vertical asymptotes: They exist when there are "poles", places where the denominator is zero.

$$\sqrt{16x^4-81}=0\to(4x^2+9)(4x^2-9)=0\to(4x^2+9)(2x-3)(2x+3)=0$$

There are two possibilities: x=3/2 and x=-3/2

Let's examine x=-3/2

$$\lim_{x\to-\frac{3}{2}^-}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}=\lim_{x\to-\frac{3}{2}^-}\frac{\frac{5}{4}}{\sqrt{16x^4-81}}=+\infty$$

$\lim_{x\to-\frac{3}{2}^+}\frac{3x^2+5x+2}{\sqrt{16x^4-81}}$ does not exist because $16x^4-81$ is negative on that side.

So, the asymptote is x=-3/2 and the function approaches $+\infty$ on the left side.

Now, let's examine x=3/2

$$\lim_{x\to\frac{3}{2}^{+}} \frac{3x^2+5x+2}{\sqrt{16x^4-81}} = \lim_{x\to\frac{3}{2}^{+}} \frac{\frac{65}{4}}{\sqrt{16x^4-81}} = +\infty$$

$\lim_{x\to\frac{3}{2}^{-}} \frac{3x^2+5x+2}{\sqrt{16x^4-81}}$ does not exist because $16x^4-81$ is negative on that side.

So, the asymptote is *x*=3/2 and the function approaches $+\infty$ on the right side.

Problem 2.- Find a function that has a vertical asymptote at *x*=-1, a horizontal asymptote at *y*=2, a hole for *x*=2 and an *x* intercept of 4.

Solution: We will try to build the function step by step. To have a vertical asymptote *x*=-1 we need a factor *x*+1 in the denominator, for example

$$f(x) = \frac{c}{x+1}$$

Where c is any constant (but not zero).

Next, we want a horizontal asymptote at $y = 2$. We notice that the function that we created already has limits 0 at $+\infty$ and $-\infty$, so we could just add 2 to the function to have the $y = 2$ asymptote.

$$f(x) = 2 + \frac{c}{x+1}$$

Next, the function should have a hole at *x*=2, we can do that by using the factor $\frac{x-2}{x-2}$, which is 1 everywhere except that it has a hole at *x*=2.

$$f(x) = \left(2 + \frac{c}{x+1}\right)\left(\frac{x-2}{x-2}\right)$$

Finally, we want an *x* intercept of 4, which means that the function should pass through (4,0). That means

$$f(4) = 0 \rightarrow \left(2 + \frac{c}{4+1}\right)\left(\frac{4-2}{4-2}\right) = 0 \rightarrow c = -10$$

So the function we found is

$$f(x) = \left(2 - \frac{5}{x+1}\right)\left(\frac{x-2}{x-2}\right)$$

$$f(x)=\frac{2x^2-7x+6}{x^2-x-2}$$