Calculus

Continuity

Problem 1.- Given the function defined as follows:

$$h(x)=\begin{cases} cx^3 & x<1 \\ 6 & x=1 \\ -x^2+mx & x>1 \end{cases}$$

Is it possible to find *c* and *m* to make *h* continuous at *x* = 6? If so, find the values.

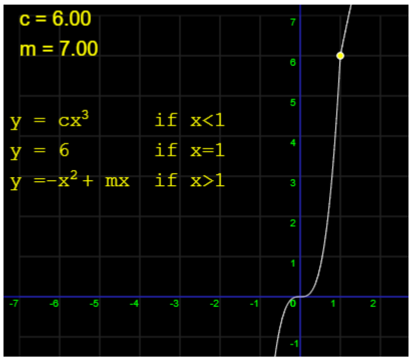
Solution: The value of *h* at *x* = 1 is 6. To be continuous, the left and right limits must be 6 as well, so:

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} cx^3 = c$$

Which means that *c* must be 6.

$$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} -x^2+mx = -1+m$$

Which means that *m* must be 7.



Problem 2.- Prove that the function is continuous on the left of *x* = 2, but not on the right.

$$f(x)=\begin{cases} x^2 & if\ x>2 \\ x+1 & if\ x\le 2 \end{cases}$$

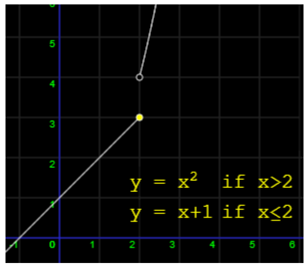
Solution: We notice that the value of *f* at *x* = 2 is *x*+1 = 3 and the limits are

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+1 = 2+1 = 3$$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 2^2 = 4$

Because the limit on the left is equal to the value of the function at x=2 then it is continuous on the left.

Because the limit on the right is not equal to the value of the function at x=2 then it is not continuous on the right.



Problem 3.- If f and g are continuous with $f(13)=4$ and $\lim_{x\to 13}\left[2f(x)-g(x)\right]=13$, find $g(13)$

Solution: Since the functions are continuous, the limit at x=13 should be equal to the value of the functions at x=13, so the equation of the limits indicates

$$\lim_{x\to 13}\left[2f(x)-g(x)\right]=13 \to 2f(13)-g(13)=13$$

And solving for g(13) we get

$$2\times 4 - g(13) = 13 \to g(13) = -5$$

Problem 4.- Find the intervals where the function f is continuous

$$f(x) = \frac{1}{x} + \frac{8\sqrt{x}}{(x-9)^2}$$

Solution: First, we notice that the domain of the function is limited to nonnegative numbers due to the square root of x, that means $x \geq 0$. Additionally, the denominator of a fraction cannot be zero, so $x \neq 0$ and $x \neq 9$.
Other than those restrictions, f is continuous in its domain, so the intervals where the function is continuous are

$\langle 0, 9\rangle$ and $\langle 9, \infty\rangle$

Problem 5.- Using the intermediate value theorem, demonstrate that a solution exists x>0 for the following equation

$$3^x = \frac{9}{x}$$

Solution: We can observe the function *f*, defined as follows:

$$f(x) = 3^x - \frac{9}{x}$$

We notice that this function is continuous in the range $\langle 0, \infty \rangle$.

In addition,

$$f(1) = 3^1 - \frac{9}{1} = -6$$

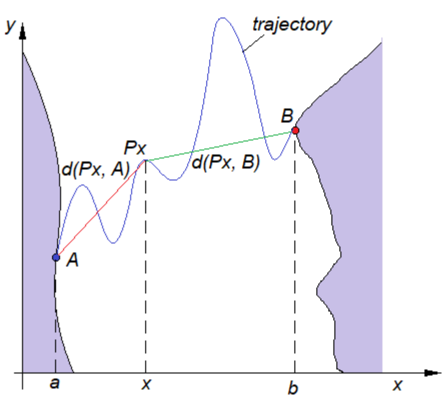
and

$$f(2) = 3^2 - \frac{9}{2} = 4.5$$

So, $f(1) < 0 < f(2)$

Due to the intermediate value theorem there must be a number *x* in the interval $\langle 1, 2 \rangle$ where the function has the value zero, and that value solves the original equation.

Problem 6.- A person crosses a pass following the trajectory shown in the graph. Show that there must be a value of x, such that the distance from P_x to *A* and to *B* are the same.



Solution: We notice from the figure that the distances *d(Px, A)* and *d(Px, B)* are continuous functions of *x*. We define:

$$f(x) = d(P_x, A) - d(P_x, B)$$

We also notice that

$$f(a) = -d(A, B)$$

And

$$f(b) = d(A, B)$$

So, because $f(b) > 0 > f(a)$ there must be a value $b>x>a$, such that $f(x) = 0$

Graphically this position can be found by drawing a normal line in the middle of the straight line from A to B and finding the intersection with the trajectory. This is because all the points on the normal line are equidistant from A and B.