Calculus

Derivatives

Problem 1.- If the following limit represents a derivative of a function *f* at a point *a*, find the equation of *f* and the value of *a*.

a) $\lim_{h\to 0}\dfrac{(3+h)^2-9}{h}$

b) $\lim_{x\to 1}\dfrac{2^x-2}{x-1}$

c) $\lim_{x\to 3}\dfrac{(x+1)^{3/2}-8}{x-3}$

d) $\lim_{h\to 0}\dfrac{\sin(\pi(2+h))-0}{h}$

Solution:

a) The definition of derivative of a function *f* at a point *a* can be written as a limit as follows

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Next, we can identify in the given equation that

$f(a+h)=(3+h)^2$

Substituting $a+h=x$ we get the equation of *f*

$f(x)=x^2$

Also, from the give equation

$a+h=3+h$

Which means that $a = 3$

b) In this case we need to be careful because variable *x* is not the "differential", we need to change variables to really have the differential. We make the substitution $h=x-1$

Which means that $x=h+1$, and the limit becomes

$$\lim_{h\to 0}\frac{2^{h+1}-2}{h}$$

Now we can identify the function *f* and the value *a*

$f(a+h) = 2^{h+1}$ which yields $f(x) = 2^x$ and $a = 1$

c) Similar to part (b), here we need to substitute $h = x - 3$, which means that $x = 3 + h$, then the limit becomes

$$\lim_{h\to 0} \frac{(h+3+1)^{3/2} - 8}{h} = \lim_{h\to 0} \frac{(h+4)^{3/2} - 8}{h}$$

Identifying the function *f* and *a* we get

$f(x) = x^{3/2}$ and $a = 4$

d) Since the limit is already in the form of a derivative definition, we can identify directly

$f(a+h) = \sin(\pi(2+h))$, which gives

$f(x) = \sin(\pi x)$ and $a = 2$

Problem 2.- Using the limit definition of derivative, find the instantaneous velocity of a car whose position s is given by the equation $s = 3t^2 + 13t$, at t = 20.

Solution: Using the definition

$$v(20) = \lim_{h\to 0} \frac{3(20+h)^2 + 13(20+h) - 3(20)^2 - 13(20)}{h}$$

$$v(20) = \lim_{h\to 0} \frac{3(400 + 40h + h^2) + 13(20+h) - 3(400) - 13(20)}{h}$$

$$v(20) = \lim_{h\to 0} \frac{133h + 3h^2}{h} = \lim_{h\to 0} 133 + 3h = 133$$

Problem 3.- Find the equation for the tangent line to the function $y = 5x - x^2$ at point (2, 6) using the limit definition of derivative.

Solution: Since we already have the tangent point, we just need the slope to get the tangent line.

$$slope = m = \lim_{h\to 0} \frac{5(2+h) - (2+h)^2 - 6}{h}$$

$$\lim_{h\to 0}\frac{10+5h-(4+4h+h^2)-6}{h}$$

$$\lim_{h\to 0}\frac{h-h^2}{h}=\lim_{h\to 0}1-h=1$$

Then the equation of the tangent is $y-6=1(x-2)$, which can be simplified to

$y=x+4$

Problem 4.- A company studies the cost of an advertisement campaign vs. an approval rating finding the following results:

Cost ($)	1000	2000	3000	3500	3600	3800	4000	5000
Approval (%)	32	33	46	55	61	65	69	70

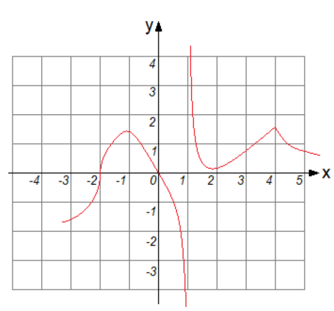
Assuming that the approval A is a function of the cost c, estimate the value of A'(3500) and interpret its meaning and units.

Solution: We can estimate the derivative using the slope of the secant:

$$A'=\frac{A(3600)-A(3000)}{3600-3000}=\frac{61-46}{600}=0.025$$

The units are percentage points per dollar. One way to interpret the result is how many percentage points of approval we gain by each dollar spent in the production value.

Problem 5.- Given the graph of a function below, indicate the places where the derivative does not exist and why.



Solution:

x=4, here the function seems to have two different slopes on the left and right, so the derivative is not defined there.

x=1, there is a vertical asymptote, so the function has no derivative there.

x=-1, there function seems to be vertical at that point, if that is the case it does not have derivative there.

Problem 6.- Find numbers a, b, and c, such that the function $f(x) = ax^2 + bx + c$ has x-intercepts at (0,0) and (5,0), and a tangent line with slope 1 where x=2.

Solution: To have intercept (0,0) it needs to be a solution of the function, so we can write

$$f(0) = a\times 0^2 + b\times 0 + c = c = 0$$

That gives us the value of c, which is 0. Next, to have an intercept at (5,0):

$$f(5) = a\times 5^2 + b\times 5 + c = 25a + 5b = 0$$

We have an equation for a and b. Finally the derivative at $x = 2$ is 1, so:

$$f'(x) = 2ax + b \qquad \text{so} \qquad f'(2) = 2a(2) + b = 4a + b = 1$$

Which gives us a second equation of a and b. We can put them together:

$$5a + b = 0$$
$$4a + b = 1$$

Solving we get $a = -1$ and $b = 5$

Problem 7.- Find the derivative of

$$r(x) = \frac{xe^x + 6x^2}{5}$$

Solution: $r'(x) = \dfrac{xe^x + e^x + 12x}{5}$

Problem 8.- Knowing

x	*f(x)*	*f'(x)*	*g(x)*	*g'(x)*
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	

Find

a) If $h(x) = e^x f(x)$, find $h'(0)$

b) If $n(x) = \dfrac{1}{f(x)}$, find $n'(1)$

Solution:

a) To find the derivative we use the product rule

$$h'(x) = (e^x)' f(x) + e^x (f(x))' = e^x f(x) + e^x f'(x)$$

And now we replace the values from the table

$$h'(0) = e^0 f(0) + e^0 f'(0) = 1 \times 5 + 1 \times 9 = 14$$

b) $n(x) = \dfrac{1}{f(x)}$

To find $n'(x)$ we use the quotient rule

$$n'(x) = \frac{(1)' f(x) - 1(f(x))'}{(f(x))^2} = \frac{-f'(x)}{(f(x))^2}$$

Then replace the values from the table

$$n'(1) = \frac{-f'(1)}{(f(1))^2} = \frac{-(-3)}{(3)^2} = \frac{1}{3}$$