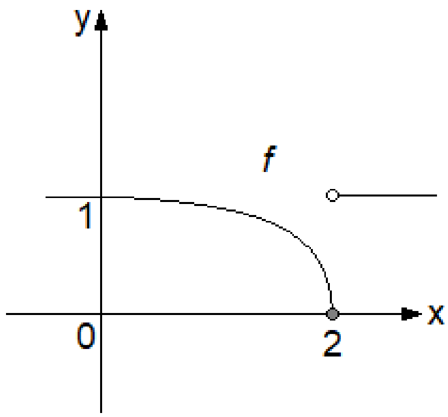
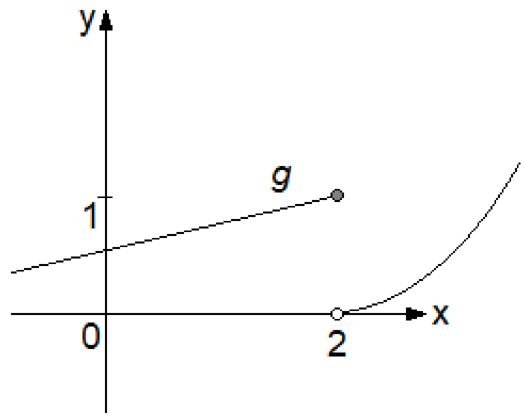
Calculus

Limits

Problem 1.- Let us define the functions *f* and *g* graphically, and *h* and *j* by the rules below.





$$h(x) = \frac{x^2 - 4}{x - 2} \qquad j(x) = \begin{cases} 1 \text{ if } x < 2 \\ 0 \text{ if } x \geq 2 \end{cases}$$

Then, find the following limits if they exist or explain why they do not.

$i)\ \lim_{x \to 2} f(x)$
$ii)\ \lim_{x \to 2} g(x)$
$iii)\ \lim_{x \to 2} h(x)$
$iv)\ \lim_{x \to 2} j(x)$
$v)\ \lim_{x \to 2} [g(x) + h(x)]$
$vi)\ \lim_{x \to 2} [f(x) + j(x)]$
$vii)\ \lim_{x \to 2} [f(x) g(x)]$
$viii)\ \lim_{x \to 2} [f(x) j(x)]$

Solution:
i) Since the limits of *f* on the right and on the left of *x*=2 are different

$$\lim_{x \to 2^-} f(x) = 0$$
$$\lim_{x \to 2^+} f(x) = 1$$

The limit does not exist

ii) Since the limits of *g* on the right and on the left of *x*=2 are different

$$\lim_{x \to 2^-} g(x) = 1$$
$$\lim_{x \to 2^+} g(x) = 0$$

The limit does not exist

iii) Before taking the limit, we notice that if $x \neq 2$, where the function *h* is undefined, it can be simplified as follows

$$h(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2$$

And here the limit is

$$\lim_{x\to 2} h(x) = x + 2 = 4$$

iv) Since the limits of j on the right and on the left of $x = 2$ are different
$\lim_{x\to 2^-} j(x) = 1$
$\lim_{x\to 2^+} j(x) = 0$
The limit does not exist

v) In this case the limits of the left and right are different, so the limit does not exist

$$\lim_{x\to 2^-}[g(x) + h(x)] = 1 + 4 = 5$$
$$\lim_{x\to 2^+}[g(x) + h(x)] = 0 + 4 = 4$$

vi) In this case the limits of the left and right are the same, so the limit does exist and it is 1.

$$\lim_{x\to 2^-}[f(x) + g(x)] = 0 + 1 = 1$$
$$\lim_{x\to 2^+}[f(x) + g(x)] = 1 + 0 = 1$$

We also notice that the value of $[f(x) + g(x)]$ at x=2 is 0, but that does not concern us when dealing with the limits.

vii) In this case the limits of the left and right are the same, so the limit does exist and it is 0.

$$\lim_{x\to 2^-}[f(x)g(x)] = 0 \times 1 = 0$$
$$\lim_{x\to 2^+}[f(x)g(x)] = 1 \times 0 = 0$$

viii) In this case the limits of the left and right are the same, so the limit does exist and it is 0.

$$\lim_{x\to 2^-}[f(x)j(x)] = 0 \times 1 = 0$$
$$\lim_{x\to 2^+}[f(x)j(x)] = 1 \times 0 = 0$$

Problem 2.- Evaluate the following limit or indicate why it does not exist.

$$\lim_{x\to 2} \frac{x^2 - 4}{3x^2 - 2x - 8}$$

Solution: We notice that at *x*=2 the fraction is undefined because it has the form 0/0, but if we factor terms and simplify, we get

$$\frac{x^2-4}{3x^2-2x-8}=\frac{(x-2)(x+2)}{(x-2)(3x+4)}=\frac{x+2}{3x+4}$$

Notice that this can be done as long as $x \neq 2$, so the limit is

$$\lim_{x\to 2}\frac{x^2-4}{3x^2-2x-8}=\lim_{x\to 2}\frac{x+2}{3x+4}=\frac{2+2}{3\times 2+4}=\frac{4}{10}=0.4$$

Problem 3.- Evaluate the following limit or indicate why it does not exist.

$$\lim_{x\to 2}\frac{x^2-4}{3x^2-2x-8}$$

Solution: We notice that at *x*=2 the fraction is undefined because it has the form 0/0, but if we factor terms and simplify, we get

$$\frac{x^2-4}{3x^2-2x-8}=\frac{(x-2)(x+2)}{(x-2)(3x+4)}=\frac{x+2}{3x+4}$$

Problem 4.- Find the value of c so that the limit $\lim_{x\to 0} f(x)$ exists

$$f(x)=\begin{cases} x+c \ if \ x<0 \\ 9-x^2 \ if \ x\geq 0 \end{cases}$$

Solution: We notice that the left and right limits are

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x+c=c$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} 9-x^2=9$$

They must be equal, so c = 9.

at *x*=2 the fraction is undefined because it has the form 0/0, but if we factor terms and simplify, we get:

$$\frac{x^2-4}{3x^2-2x-8}=\frac{(x-2)(x+2)}{(x-2)(3x+4)}=\frac{x+2}{3x+4}$$

Problem 5.- Use the squeeze theorem to find the limit

$$\lim_{x\to 0} x\sin\left(\frac{e}{x}\right)$$

Solution: We notice that the sine function is bound $-1 \leq \sin\left(\frac{e}{x}\right) \leq 1$

So, the function given is also bound:

$$-|x| \leq x\sin\left(\frac{e}{x}\right) \leq |x|$$

And since the limits of the lower bound and upper bound are both the same (zero) then the limit of the function is zero

$$\lim_{x\to 0} x\sin\left(\frac{e}{x}\right) = 0$$

Problem 6.- Calculate

a) $\lim_{x\to 5} \frac{(x-3)^4 - 16}{x-5}$

b) $\lim_{x\to 9} \frac{x^2 - 81}{3-\sqrt{x}}$

Solution:

a) $\lim_{x\to 5} \frac{(x-3)^4 - 16}{x-5} = \lim_{x\to 5} \frac{\left((x-3)^2 - 4\right)\left((x-3)^2 + 4\right)}{x-5}$

$$\lim_{x\to 5} \frac{\left((x-3)+4\right)\left((x-3)-2\right)\left((x-3)^2+4\right)}{x-5} = \lim_{x\to 5}(x-1)\left((x-3)^2+4\right) = (4)(8) = 32$$

b) $\lim_{x\to 9} \frac{x^2 - 81}{3-\sqrt{x}} = \lim_{x\to 9} \frac{(x-9)(x+9)}{3-\sqrt{x}} \frac{(3+\sqrt{x})}{(3+\sqrt{x})} = \lim_{x\to 9} \frac{(x-9)(x+9)(3+\sqrt{x})}{9-x}$

$$= \lim_{x\to 9} -(x+9)(3+\sqrt{x}) = -162$$

Problem 7.- Calculate the values of a and b, so the limit exists, and it is equal to 1.

$$\lim_{x\to 0} \frac{\sqrt{ax+b} - 2}{x}$$

Solution: We multiply numerator and denominator by the conjugate to get rid of the square root in the numerator:

$$= \lim_{x\to 0} \frac{\sqrt{ax+b} - 2}{x} \frac{\sqrt{ax+b} + 2}{\sqrt{ax+b} + 2}$$

$$= \lim_{x\to 0} \frac{ax+b-4}{x} \frac{1}{\sqrt{ax+b} + 2}$$

For the limit to exist we need to cancel the x in the denominator, so b has to be 4, then:

$$= \lim_{x\to 0} \frac{ax}{x} \frac{1}{\sqrt{ax+4}+2} = \lim_{x\to 0} \frac{a}{\sqrt{4}+2} = \lim_{x\to 0} \frac{a}{4}$$

And if the limit is 1, then $a = 4$ as well.

Problem 8.- Find the limit.

$$\lim_{y\to -\infty} y+\sqrt{y^2+5y}$$

Solution:

$$= \lim_{y\to -\infty} = y+\sqrt{y^2+5y}\,\frac{y-\sqrt{y^2+5y}}{y-\sqrt{y^2+5y}}$$

$$= \lim_{y\to -\infty} \frac{y^2-y^2-5y}{y-\sqrt{y^2+5y}} = \lim_{y\to -\infty} \frac{-5y}{y-\sqrt{y^2+5y}} \frac{1/y}{1/y}$$

The tricky part is the next step, because

$$-5y/y = -5$$
$$y/y = 1$$

But $-\sqrt{y^2+5y}/y = +\sqrt{\left(y^2+5y\right)/y^2}$

The change in sign is because $y = -\sqrt{y^2}$, so then:

$$= \lim_{y\to -\infty} \frac{-5}{1+\sqrt{1+5/y}} = -5/2$$

Problem 9.- Find the limit.

$$\lim_{y\to \infty} \arctan\left(\frac{1-x^2}{x+4}\right)$$

Solution:

$$= \lim_{y\to \infty} \arctan\left(\frac{1-x^2}{x+4}\frac{1/x}{1/x}\right)$$

$$= \lim_{y\to \infty} \arctan\left(\frac{1/x-x}{1+4/x}\right) = \lim_{y\to \infty} \arctan\left(\frac{0-x}{1+0}\right)$$

$$= \lim_{y\to \infty} \arctan\left(-x\right) = -\frac{\pi}{2}$$

Problem 10.- Find the limit.

$$\lim_{y\to\infty} \frac{\cos x}{\ln x}$$

Solution: Notice that for x>1 (which makes ln x>0)

$$\frac{1}{\ln x} \geq \frac{\cos x}{\ln x} \geq \frac{-1}{\ln x}$$

And $\lim_{y\to\infty} \frac{1}{\ln x} = \lim_{y\to\infty} \frac{-1}{\ln x} = 0$

So, using the squeeze theorem $\lim_{y\to\infty} \frac{\cos x}{\ln x} = 0$