Calculus

Derivatives of trigonometric functions

Problem 1.- Find the derivative of each function

a) $f(x) = 2x^3 \sin x$

b) $f(\theta) = e^\theta(\cos\theta - \tan\theta)$

c) $g(x) = x^2 e^x \sec x$

d) $h(x) = \csc x \cot x$

e) $f(x) = \dfrac{(x^2-1)\sin x}{\sin x + 1}$

f) $g(x) = \dfrac{1}{\sec x \csc x}$

Solution: We apply the rules of differentiation directly, but convert the secant, cosecant and cotangent to sines and cosines for convenience in d and f.

a) $\dfrac{d}{dx} 2x^3 \sin x = 6x^2 \sin x + 2x^3 \cos x$

b) $\dfrac{d}{d\theta} e^\theta(\cos\theta - \tan\theta) = e^\theta(\cos\theta - \tan\theta) + e^\theta(-\sin\theta - \sec^2\theta)$

$= e^\theta(\cos\theta - \tan\theta - \sin\theta - \sec^2\theta)$

c) $\dfrac{d}{dx} x^2 e^x \sec x = 2xe^x \sec x + x^2 e^x \sec x + x^2 e^x \sec x \tan x$

$= xe^x \sec x(2 + x + x\tan x)$

d) $\dfrac{d}{dx} \csc x \cot x = \dfrac{d}{dx} \dfrac{\cos x}{\sin^2 x} = \dfrac{\sin^2 x - 2}{\sin^3 x}$

e) $\dfrac{d}{dx} \dfrac{(x^2-1)\sin x}{\sin x + 1}$ We use the quotient rule $\left(f/g\right)' = \dfrac{f'g - fg'}{g^2}$

First, we get the derivative of the numerator $f' = 2x\sin x + (x^2-1)\cos x$

And the derivative of the denominator $g' = \cos x$

$$\frac{d}{dx}\frac{(x^2-1)\sin x}{\sin x + 1} = \frac{(2x\sin x + (x^2-1)\cos x)(\sin x + 1) - (x^2-1)\sin x \cos x}{(\sin x + 1)^2}$$

$$= \frac{2x\sin^2 x + 2x\sin x + (x^2 - 1)\cos x}{(\sin x + 1)^2}$$

f) It is simpler if we write g with sines and cosines

$$g(x) = \frac{1}{\sec x \csc x} = \sin x \cos x$$

Then $\frac{d}{dx} g(x) = \cos^2 x - \sin^2 x$

That we recognize as $\cos 2x$

Problem 2.- Find the tangent line of the function $f(x) = 3x\sin x$ at $x = \frac{\pi}{3}$.

Solution: The value of the function at $x = \frac{\pi}{3}$ is

$$f(\pi/3) = 3\frac{\pi}{3}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{2},$$

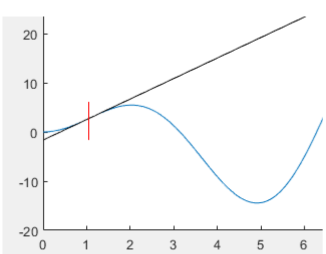
So, the tangent line passes through the point $\left(\frac{\pi}{3}, \frac{\pi\sqrt{3}}{2}\right)$.

The derivative of the function at $x = \frac{\pi}{3}$ is

$$f'(\pi/3) = 3\sin\frac{\pi}{3} + 3\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{3\sqrt{3} + \pi}{2}$$

The equation of the tangent is

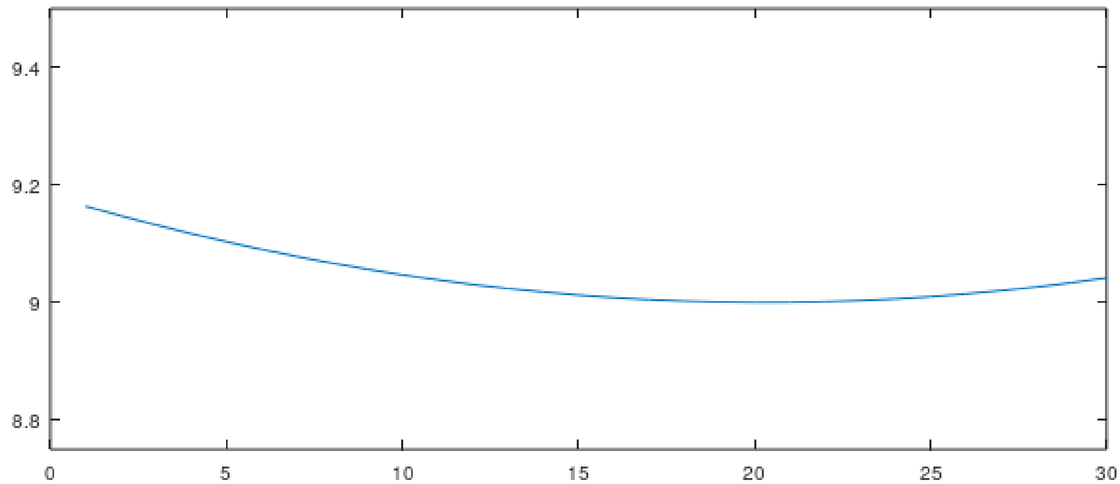
$$y - \frac{\pi\sqrt{3}}{2} = \frac{3\sqrt{3} + \pi}{2}\left(x - \frac{\pi}{3}\right)$$



Problem 3.- The number of hours of daylight in a city in the northern hemisphere is given by the equation

$$D(t) = 12 - 3\cos(0.0172(t + 11))$$

Where t is the number of the days in the year (January 1st is 1). Here is an example on one month:



a) On what day of this month is the minimum reached? On what day is the maximum?

b) What month does the graph represent?

c) For what month will the graph appear as this, but upside down?

d) How rapidly are we gaining daylight 90 days after the minimum occurs?

e) A report says that starting at day 68 after the minimum we gained 1 hour 35 minutes in 31 days. Based on your calculations, how accurate is this report?

Solution: I notice that the function is periodic, with period one year or 365 days, because $365 \times 0.0172 = 6.28 = 2\pi$

a) By inspecting the graph, it seems that the minimum for that month occurs on day 20, and the maximum occurs on day 1.

b) My experience shows that the shortest days in the northern hemisphere happen in December. But also, mathematically the function D has its minimum when the cosine is equal to 1, which happens when t+11 is zero or t=-11 or t=365-11, so 11 days before new year. That is December.

c) Since the function is a cosine, it should have the maximum half a period later than the minimum. That is half a year later, in June.

d) Taking the derivative, we get

$$D'(t) = 0.0516\sin(0.0172(t+11))$$

We already found that the minimum occurs at t+11=-0, if we add 90 days we get t+11 = 90

$$D'(t) = 0.0516\sin(0.0172(90)) = 0.0516 \text{ hours/day}$$

e) If the report starts at t + 11 = 68 and ends at 68 + 31 = 99 days then the change should be close to the derivative in the middle of that period t + 11 = 84, so

$$D'(t) = 0.0516\sin(0.0172(84)) = 0.0512 \text{ hours/day}$$

And in 31 days we have

$$D'(t)\times 31 = 0.0512\times 31 = 1.59 hours$$

The 0.59 hours in minutes is 0.59×60 = 35 minutes

In conclusion, the report is accurate.

Problem 4.- Find

$$\frac{d^{45}}{dx^{45}} x^2 \cos x$$

Solution: I notice

$$\frac{d}{dx} x^2 \cos x = 2x\cos x - x^2 \sin x$$

$$\frac{d}{dx} x^2 \sin x = 2x\sin x + x^2 \cos x$$

$$\frac{d}{dx} x\cos x = \cos x - x\sin x$$

$$\frac{d}{dx} x\sin x = \sin x + x\cos x$$

$$\frac{d}{dx} \cos x = \sin x$$

$$\frac{d}{dx}\sin x = -\cos x$$

With these 6 rules we can build a table like this:

	$\sin x$	$\cos x$	$x\sin x$	$x\cos x$	$x^2\sin x$	$x^2\cos x$
f(x)	*a = 0*	*b = 0*	*c = 0*	*d = 0*	*e = 0*	*f = 1*
f'(x)	*-b + c = 0*	*a + d = 0*	*2e - d = 0*	*2f + c = 2*	*-f = -1*	*e = 0*
$f^2(x)$						
$f^3(x)$						
$f^4(x)$						

And continuing all the way to 45:

$$\frac{d^{45}}{dx^{45}}x^2\cos x = 1980\sin x + 90x\cos x - x^2\sin x$$