Calculus

Implicit differentiation

Problem 1.- Find *dy/dx* by implicit differentiation

a) $xy(2x+3y)=2$
b) $(2x+3y)^2=10$
c) $\cos xy=1-x^2$
d) $e^{xy}+\ln y^2=x$

Solution: We apply the rules of differentiation in each equation and the solve for *y'*

a) We have three factors on the left side of the equation, so we will need to apply the product rule there

$$(xy(2x+3y))'=(x'y(2x+3y))+(xy'(2x+3y))+(xy(2x+3y)')$$

Then

$$y(2x+3y)+xy'(2x+3y)+xy(2+3y')=0$$

And solving for *y'*

$$y'=-\frac{y(4x+3y)}{2x(3y+x)}$$

b) We take the derivative and solve for *y'*

$$2(2x+3y)(2+3y')=0$$
$$y'=-\frac{2}{3}$$

c) In this case we will need to use the chain rule, taking the derivative of cosine and then the derivative of the argument.

$$-\sin xy(y+xy')=-2x$$
$$y\sin xy+x\sin xyy'=2x$$
$$y'=\frac{2x-y\sin xy}{x\sin xy}$$

d) For the exponential we need to apply the chain rule. For the logarithm, we can first convert $\ln y^2$ to $2\ln y$, then, taking derivative:

$$e^{xy}(y+xy')+\frac{2}{y}y'=1$$

$$y'=\frac{y(1-ye^{xy})}{2+xye^{xy}}$$

Problem 2.- Find the equation for the tangents at the points indicated

a) $\sin(x-y)=xy$ at $(0, \pi)$
b) $x\tan^{-1}y=x^2+y$ at $(0, 0)$

Solution: We need to find *y'* at the tangent points to get the equations, which we do using implicit differentiation.

a) Taking the derivative and solving for *y'*

$$\cos(x-y)(1-y')=y+xy'$$

$$y'=\frac{\cos(x-y)-y}{x+\cos(x-y)}$$

And we evaluate this function at $(0, \pi)$

$$y'=\frac{\cos(0-\pi)-\pi}{0+\cos(0-\pi)}=\frac{-1-\pi}{0-1}=\pi+1$$

Then the equation of the tangent is

$$y=(\pi+1)x+\pi$$

b) Taking the derivative

$$\tan^{-1}y+\frac{x}{y^2+1}y'=2x+y'$$

Solving for *y'*

$$y'=\frac{(y^2+1)(2x-\tan^{-1}y)}{x-y^2-1}$$

Evaluating the derivative

$$y'=\frac{(0^2+1)(2-\tan^{-1}0)}{0-0^2-1}=0$$

So, the equation of the tangent is

$$y = 0$$

Problem 3.- Find the equation for normal line to the curve

$x^2\sqrt{y-2} = y^2 - 3x - 5$ at (1, 3)

Solution: We need to find *y'* at the indicated point and then the normal line will have slope -1/*y'*.

Taking the derivative and solving for *y'*

$$2x\sqrt{y-2} + \frac{x^2}{2\sqrt{y-2}}y' = 2yy' - 3$$
$$2\sqrt{3-2} + \frac{1^2}{2\sqrt{3-2}}y' = 2\cdot 3y' - 3$$
$$y' = 1/1.1$$

And the normal is

$$y - 3 = -1.1(x-1)$$
$$y = 4.1 - 1.1x$$

Problem 3.- Consider the solutions to the equation

$$x^{p/q} + y^{p/q} = 1$$

Where *p* is even, and *q* is odd. When *p/q*<1 the curves are called “astroids” for the resemblance to drawings of stars.

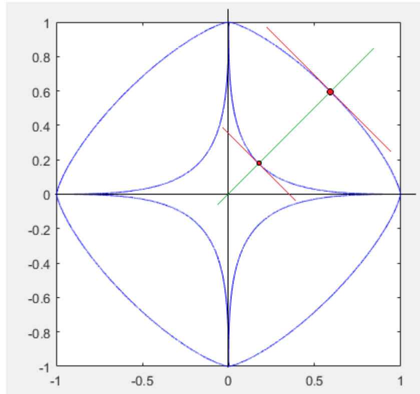
At what points are the slopes of the tangents +1 or -1, when *p/q* = 4/3 and 2/5?

Solution: Due to symmetry the slopes will be +1 or -1 when *y* = *x* or *y* = -*x* so, in the equation $x^{p/q} + y^{p/q} = 1$, if we replace $y = x$ we get

$$x^{p/q} + x^{p/q} = 1 \rightarrow x^{p/q} = 1/2$$

And solving for *x* we get $x = 2^{-q/p}$

The points are $(2^{-q/p}, 2^{-q/p}), (2^{-q/p}, -2^{-q/p}), (-2^{-q/p}, 2^{-q/p})$ *and* $(-2^{-q/p}, -2^{-q/p})$,



1
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0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1