Calculus

Linear approximation

Problem 1.- A manufacturer's cost is given by

$$C(q) = 0.1q^3 - 0.5q^2 + 500q + 200$$

What will be the change in cost if the quantity gets reduced from 4 units to 3.9 units?

Solution: A simple way to evaluate the change would be to get the difference

$$\Delta C = C(4) - C(3.9)$$

But we can also use the linear approximation

$$C(q) \approx C(4) + (q-4)C'(4)$$

Then

$$C(3.9) - C(4) \approx \{C(4) + (3.9-4)C'(4)\} - \{C(4) + (4-4)C'(4)\}$$
$$\Delta C \approx (3.9-4)C'(4)$$

And
$$C'(q) = 0.3q^2 - q + 500$$
$$C'(4) = 0.3\times 4^2 - 4 + 500 = 500.8$$

So $\quad \Delta C \approx (3.9-4)C'(4) = -0.1\times 500.8 = -50.08$

Problem 2.- If the derivative of a function is given by

$$f'(x) = 3 - \frac{x^2}{3}$$

And $f(2)=4$, estimate the value of $f(1.98)$ and $f(2.02)$

Are your approximations overestimations or underestimations?

What happens if $f(3)=7$ and you want to estimate $f(2.98)$ and $f(3.02)$?

Solution: Using the linear approximation close to $x=2$ we get

$$f(x) \approx f(2) + (x-2)f'(2)$$

In the problem $f'(2) = 3 - \frac{2^2}{3} = 1.67$

So $f(1.98) \approx f(2) + (1.98 - 2) \times 1.67 = 3.97$

And $f(2.02) \approx f(2) + (2.02 - 2) \times 1.67 = 4.03$

Since the derivative is decreasing close to *x*=2, the function is concave downwards, which means that using a linear approximation will overestimate the values in both cases.

Since $f'(3) = 0$, the linear approximation gives us

$f(2.98) \approx f(3) = 7$ and

$f(3.02) \approx f(3) = 7$

So, to get a better approximation we would need a ***quadratic*** approximation, not linear.

Problem 3.- Using the linear approximation, estimate the value of

$$\sqrt[4]{4100} + \sqrt[3]{4100} + 3\sqrt{4100}$$

Solution: We notice that 4100 is close to 2^{12}=4096, and we define the function

$$f(x) = \sqrt[4]{x} + \sqrt[3]{x} + 3\sqrt{x}$$

So

$$f(4096) = \sqrt[4]{2^{12}} + \sqrt[3]{2^{12}} + 3\sqrt{2^{12}}$$
$$f(4096) = 2^3 + 2^4 + 3 \times 2^6$$
$$f(4096) = 216$$

Also $f'(x) = \frac{1}{4}x^{-3/4} + \frac{1}{3}x^{-2/3} + 3\frac{1}{2}x^{-1/2}$

$$f'(4096) = \frac{1}{4}(2^{12})^{-3/4} + \frac{1}{3}(2^{12})^{-2/3} + 3\frac{1}{2}(2^{12})^{-1/2}$$
$$f'(4096) = \frac{1}{4}(2^{-9}) + \frac{1}{3}(2^{-8}) + 3\frac{1}{2}(2^{-6})$$

For the value of 4100, using the linear approximation:

$$f(4100) \approx f(4096) + (4100 - 4096) f'(4096) \approx 216.1$$