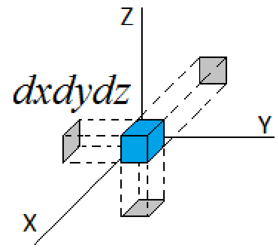
Calculus

Cartesian Coordinates

To specify a point in three-dimensional space, the most direct form is to define three planes perpendicular to each other and specify the distances to them as coordinates (x, y, z).

For volume integrals, we use a differential given by:

$dV = dxdydz$



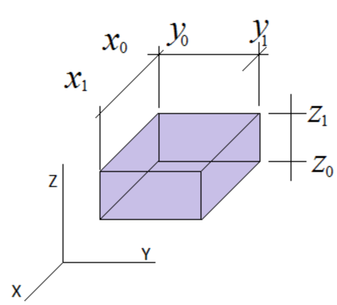
In Cartesian coordinates the unit vectors along the three axes are:

$\hat{x} = (1,0,0)$
$\hat{y} = (0,1,0)$
$\hat{z} = (0,0,1)$

Setting up integrals in Cartesian coordinates

The simplest case is a rectangular prism in which the volume to integrate is a box with sides perpendicular to the axes X, Y and Z, whose limits are:

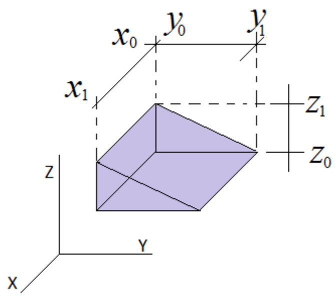
$x \in [x_0, x_1]$
$y \in [y_0, y_1]$
$z \in [z_0, z_1]$



The differential of volume is $dV = dxdydz$ and we can use the limits directly in the triple integral corresponding to the limits of the box.

$$Q = \int_V \rho dV = \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \rho dx dy dz$$

Another example is the integral over a volume whose sides are oblique, for example the volume in the figure below:



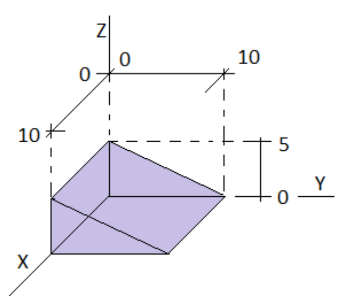
We see that in this case the limits of x are fixed throughout the whole volume, but y and z depend on each other. We can determine the limits of y for a given value of z as follows:

$$y \in \left[y_0, y_1 - \frac{y_1 - y_0}{z_1 - z_0}(z - z_0) \right]$$

What we need to do is take z as a constant when integrating over y, but set the limits in the integral functions of z as follows:

$$Q = \int_V \rho dV = \int_{z_0}^{z_1} \int_{y_0}^{y_1 - \frac{y_1 - y_0}{z_1 - z_0}(z - z_0)} \int_{x_0}^{x_1} \rho dx dy dz$$

Example: Calculate the total charge in the volume shown in the figure.



The density of charge is given by the equation:

$$\rho = (3x^2 + y)$$

Solution: We observe that if z is considered constant, the limits for y will be:

$$y_0 = 0,\ y_1 = 10 - 2z$$

Then the integral becomes:

$$Q = \int_0^5 \int_0^{10-2z} \int_0^{10} (3x^2 + y)dxdydz$$

We integrate over x

$$Q = \int_0^5 \int_0^{10-2z} \left(x^3 + xy\right)\Big|_0^{10} dydz = \int_0^5 \int_0^{10-2z} \left(1000 + 10y\right)dydz$$

Then, we integrate over y, but keeping the limits stablished above:

$$Q = \int_0^5 \left(1000y + 5y^2\right)\Big|_0^{10-2z} dz = \int_0^5 \left(1000(10-2z) + 5(10-2z)^2\right)dz$$

$$Q = \int_0^5 \left(10500 - 2200z + 20z^2\right)dz$$

Finally, we integrate over z:

$$Q = \left(10500z - 1100z^2 + 20\frac{z^3}{3}\right)\Bigg|_0^5 = \mathbf{25,800}$$