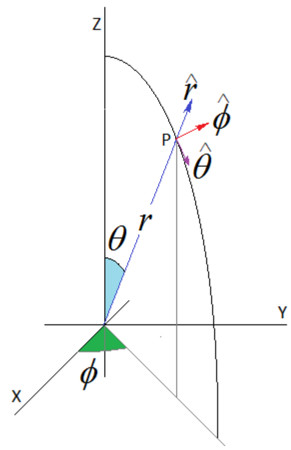
Calculus

Spherical Coordinates

There are several reasons why we find spherical symmetry in nature such as planets, stars, raindrops, atoms from noble gases, etc. There are also a variety of human creations that possess spherical symmetry such as ball bearings, balls for sports, balls for milling. In these cases it might be convenient to represent the coordinates with a radius r and two angles, ϕ and θ.



The relation between the Cartesian and Spherical systems of coordinates is given by the equations:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

The unitary vectors in the direction of variation of r, ϕ and θ, are called $\hat{r}$, $\hat{\phi}$ and $\hat{\theta}$ and they are perpendicular to each other.

For volume integrals in spherical coordinates we use the differential:

$$dV = r^2 \sin\theta d\phi d\theta dr$$

Problem 1.- Calculate the total electric charge in a dielectric sphere with radius $R = 2$, where the density of charge is given by:

$$\rho = (x^2 + y^2 + z^2)z^2$$

Solution: We notice that the density can be written in spherical coordinates:

$$\rho = r^4 \cos^2 \theta$$

The integral becomes

$$Q = \int_0^{\pi}\int_0^{2}\int_0^{2\pi} (r^4 \cos^2 \theta) r^2 \sin\theta d\phi dr d\theta$$

We integrate in order

$$Q = 2\pi \int_0^{\pi}\int_0^{2} (r^4 \cos^2 \theta) r^2 \sin\theta dr d\theta$$

$$Q = 2\pi \frac{2^7}{7} \int_0^{\pi} \cos^2 \theta \sin\theta d\theta$$

$$Q = 2\pi \frac{2^7}{7} \frac{2}{3}$$

= **76.6**