Electromagnetism

Curl

Problem 1.- Find the curl of the following vector:

$\vec{v} = (-y, x, z)$

Solution:

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 2\hat{z}$$

Problem 2.- Find the curl of the following vector:

$$\vec{v} = \left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \right)$$

Solution:

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \dfrac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \dfrac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix} =$$

$$\nabla\times\vec{v} = \hat{x}\left[\frac{\partial}{\partial y}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} \right]$$

$$-\hat{y}\left[\frac{\partial}{\partial x}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} \right]$$

$$+\hat{z}\left[\frac{\partial}{\partial x}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial y}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} \right]$$

$$\nabla\times\vec{v}=\hat{x}\left[\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{-3zy}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$-\hat{y}\left[\frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$+\hat{z}\left[\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$\nabla\times\vec{v}=(0,0,0)$$

Problem 3.- Prove the following identity:

$$\nabla\times\left(\nabla\times\vec{v}\right)\equiv\nabla\left(\nabla\cdot\vec{v}\right)-\nabla^2\vec{v}$$

Solution: First, we calculate the left side of the equation:

$$\nabla\times\left(\nabla\times\vec{v}\right)=\nabla\times\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z},-\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z},\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

$$\nabla\times\left(\nabla\times\vec{v}\right)=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial v_z}{\partial y}-\dfrac{\partial v_y}{\partial z} & -\dfrac{\partial v_z}{\partial x}+\dfrac{\partial v_x}{\partial z} & \dfrac{\partial v_y}{\partial x}-\dfrac{\partial v_x}{\partial y}\end{vmatrix}$$

$$\nabla\times\left(\nabla\times\vec{v}\right)=$$

$$\hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(-\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\right]+\hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]+$$

$$\hat{z}\left[\frac{\partial}{\partial x}\left(-\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\right]$$

$$\nabla\times\left(\nabla\times\vec{v}\right)=$$

$$\hat{x}\left[\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2}\right]+\hat{y}\left[\frac{\partial^2 v_z}{\partial y\partial z}+\frac{\partial^2 v_x}{\partial x\partial y}-\frac{\partial^2 v_y}{\partial x^2}-\frac{\partial^2 v_y}{\partial z^2}\right]+$$

$$\hat{z}\left[\frac{\partial^2 v_x}{\partial x\partial z}+\frac{\partial^2 v_y}{\partial y\partial z}-\frac{\partial^2 v_z}{\partial x^2}-\frac{\partial^2 v_z}{\partial y^2}\right]$$

Now we look at the right side of the equation:

$$\nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}$$

$$\nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}=\nabla\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(v_x,v_y,v_z)$$

$$=\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z},\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial y\partial z},\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}+\frac{\partial^2 v_z}{\partial z^2}\right)-$$

$$\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2},\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2},\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

$$=\left(\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2},\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_z}{\partial y\partial z}-\frac{\partial^2 v_y}{\partial x^2}-\frac{\partial^2 v_y}{\partial z^2},\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}-\frac{\partial^2 v_z}{\partial x^2}-\frac{\partial^2 v_z}{\partial y^2}\right)$$

They are the same!