Electromagnetism

Divergence

Problem 1.- Find the divergence of the following vector:

$$\vec{v} = (xy,\ yz,\ zx)$$

Solution:

$$\nabla \cdot \vec{v} = y + z + x$$

If the three coordinates were positive, this divergence could be called the “Manhattan distance” from the origin to (x, y, z).

Problem 2.- Find the divergence of the following vector:

$$\vec{v} = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, 0 \right)$$

If this vector represents an electric field, can you say anything about where the charge is located?

Solution:

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x} \frac{x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{y}{x^2 + y^2}$$

$$\nabla \cdot \vec{v} = \frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2} + \frac{x^2 + y^2 - 2y^2}{\left(x^2 + y^2\right)^2} = 0$$

Since the divergence of this vector is zero, if it represented an electric field, the density of charge would be zero everywhere that the divergence can be calculated. However, notice that the divergence cannot be calculated on the z-axis because the denominator is zero there. We can then conclude that the charge that produces the electric field must be on the z axis, at $x = 0$, $y = 0$.

Problem 3.- Find the divergence of the following vector:

$$\vec{v} = r^2 \exp(-r^2) \sin\theta \hat{r}$$

And evaluate its volume integral inside a sphere of radius R with center at the origin of coordinates.

Solution:

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 r^2 \exp(-r^2)\sin\theta = \sin\theta \frac{1}{r^2}\left(4r^3 \exp(-r^2) - 2r^5 \exp(-r^2)\right)$$
$$\nabla \cdot \vec{v} = 2r\left(2 - r^2\right)\sin\theta \exp(-r^2)$$

And the integral in the indicated volume is:

$$\int \nabla \cdot \vec{v} = \int_0^R \int_0^\pi \int_0^{2\pi} 2r\left(2 - r^2\right)\sin\theta \exp(-r^2) r^2 \sin\theta d\phi d\theta dr$$
$$\int \nabla \cdot \vec{v} = 2\pi \int_0^R \int_0^\pi 2r\left(2 - r^2\right)\sin\theta \exp(-r^2) r^2 \sin\theta d\theta dr$$
$$\int \nabla \cdot \vec{v} = 2\pi^2 \int_0^R \left(2r^3 - r^5\right)\exp(-r^2) dr$$

Here we need the integrals:

$$\int r^3 \exp(-r^2) dr = -\frac{1}{2}\left(1 + r^2\right)\exp(-r^2)$$
$$\int r^5 \exp(-r^2) dr = \left(-1 - r^2 - \frac{1}{2}r^4\right)\exp(-r^2)$$

So, in the problem:

$$\int \nabla \cdot \vec{v} = \pi^2 r^4 \exp(-r^2)\Big|_0^R = \pi^2 R^4 \exp(-R^2)$$

Perhaps it is easier in this case to calculate the integral using the divergence theorem and converting it to a surface integral.

Problem 4.- Verify the divergence theorem for the following vector:

$$\vec{v} = r^2 \cos\theta \hat{r} + r^2 \cos\phi \hat{\theta} - r^2 \cos\theta \sin\phi \hat{\phi}$$

Applied to a volume of 1/8 of a sphere as defined by the conditions:

$$x^2 + y^2 + z^2 \le R^2$$
$$x \ge 0,\ y \ge 0,\ z \ge 0$$

Solution: We will independently calculate the volume and the surface integrals to compare.

For the volume integral, we first calculate the divergence of the vector in spherical coordinates:

$$\nabla.\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r} r^4 \cos\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta r^2 \cos\phi - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} r^2 \cos\theta \sin\phi$$

$$\nabla.\vec{v} = 4r\cos\theta + \frac{r}{\sin\theta}\cos\theta\cos\phi - \frac{r}{\sin\theta}\cos\theta\cos\phi = 4r\cos\theta$$

Next, we calculate the volume integral of this function:

$$\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} \nabla.\vec{v} r^2 \sin\theta d\phi d\theta dr = \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} 4r\cos\theta r^2 \sin\theta d\phi d\theta dr$$

$$= \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} 4r^3 \cos\theta \sin\theta d\phi d\theta dr$$

$$= \frac{\pi}{2} \int_0^R \int_0^{\pi/2} 4r^3 \cos\theta \sin\theta d\theta dr$$

$$= \frac{\pi}{4} \int_0^R 4r^3 dr$$

$$= \frac{\pi}{4} R^4$$

Now, we calculate the surface integral

$$\oint_S v.d\vec{a}$$

We divide the octant surface into 4 parts:

i) For the surface in the xz plane, where $\phi = 0$ and $d\vec{a} = -rd\theta dr\hat{\phi}$ the integral is:

$$\int_0^R \int_0^{\pi/2} \vec{v} \cdot (-\hat{\phi}) rd\theta dr = \int_0^R \int_0^{\pi/2} r^3 \cos\theta \sin\phi d\theta dr = 0$$

ii) For the surface in the xy plane, where $\theta = \pi/2$ and $d\vec{a} = rd\phi dr\hat{\theta}$ the integral is:

$$\int_0^R \int_0^{\pi/2} \vec{v} \cdot \hat{\theta} rd\phi dr = \int_0^R \int_0^{\pi/2} r^3 \cos\phi d\phi dr$$

$$= \int_0^R r^3 dr = \frac{R^4}{4}$$

iii) For the surface in the *yz* plane, where $\phi = \pi / 2$ and $d\vec{a} = rd\theta dr\hat{\phi}$ the integral is:

$$\int_0^R \int_0^{\pi/2} \vec{v} \cdot \hat{\phi} r d\theta dr = -\int_0^R \int_0^{\pi/2} r^3 \cos\theta \sin\phi d\theta dr$$

$$= -\int_0^R r^3 dr = -\frac{R^4}{4}$$

iv) For the curved surface, $r = R$ and $d\vec{a} = R^2 \sin\theta d\theta d\phi \hat{r}$ the integral is:

$$\int_0^{\pi/2} \int_0^{\pi/2} \vec{v} \cdot \hat{r} R^2 \sin\theta d\theta d\phi = \int_0^{\pi/2} \int_0^{\pi/2} r^2 \cos\theta R^2 \sin\theta d\theta d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} R^4 \cos\theta \sin\theta d\theta d\phi = \frac{\pi R^4}{2} \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

$$= \frac{\pi R^4}{4}$$

Adding all four integrals we get the same result:

$$\oint_S v.d\vec{a} = 0 + \frac{R^4}{4} - \frac{R^4}{4} + \frac{\pi R^4}{4} = \frac{\pi R^4}{4}$$

This confirms the divergence theorem for this case.