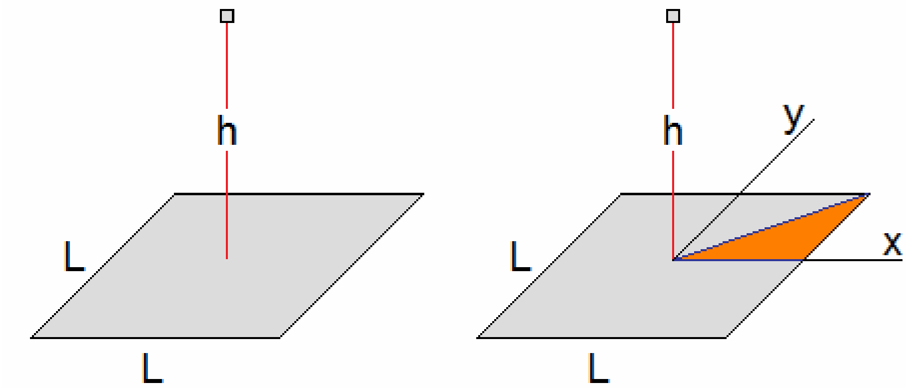
**Solid angle due to a square**

In calculating the electric field due to a square with surface charge density equal to $\sigma$ you will encounter an integral over the surface. The same integral (up to a constant factor) appears when you calculate the solid angle subtended by the square.

***Geometry***: Consider the side of the square to be L and the distance from the point of view to the center of the square to be h. The point of view is directly above the square.




We could integrate over the entire square or, to simplify the calculation just consider one triangle (as shown above) and multiply by 8.

$$S.A. = 8\int_0^{\frac{L}{2}}\int_0^{x} \frac{hdxdy}{\left(x^2+y^2+h^2\right)^{3/2}}$$

Change of variable: $y = \sqrt{x^2+h^2}\tan\theta$

$$\rightarrow dy = \sqrt{x^2+h^2}\sec^2\theta d\theta$$

$$\rightarrow \left(x^2+y^2+h^2\right)^{3/2} = \left(x^2+h^2\right)^{3/2}\sec^3\theta$$

Limits:

$$0 = \sqrt{x^2+h^2}\tan\theta \rightarrow \theta = 0$$

$$x = \sqrt{x^2+h^2}\tan\theta \rightarrow \theta = \tan^{-1}\left(\frac{x}{\sqrt{x^2+h^2}}\right)$$

so, $S.A. = 8\int_0^{\frac{L}{2}}\int_0^{\tan^{-1}\left(\frac{x}{\sqrt{x^2+h^2}}\right)} \frac{hdx\sqrt{x^2+h^2}\sec^2\theta d\theta}{\left(x^2+h^2\right)^{3/2}\sec^3\theta} = 8\int_0^{\frac{L}{2}}\int_0^{\tan^{-1}\left(\frac{x}{\sqrt{x^2+h^2}}\right)} \frac{hdx\cos\theta d\theta}{x^2+h^2}$

simplifying: $S.A. = 8\int_0^{\frac{L}{2}} \frac{h}{x^2+h^2}\frac{x}{\sqrt{2x^2+h^2}}dx$

Another change of variable: $\frac{x^2}{x^2+h^2} = \cos\theta$

$$\to x^2 = \frac{h^2\cos\theta}{1-\cos\theta} \to 2xdx = \frac{-\sin\theta}{(1-\cos\theta)^2}h^2 d\theta$$

$$\to 2x^2+h^2 = h^2\frac{1+\cos\theta}{1-\cos\theta}$$

$$\to x^2+h^2 = \frac{h^2\cos\theta}{1-\cos\theta}+h^2 = \frac{h^2}{1-\cos\theta}$$

Limits:

$$x = 0 \to \frac{0^2}{0^2+h^2} = \cos\theta \to \theta = \pi/2$$

$$x = L/2 \to \frac{L^2/4}{L^2/4+h^2} = \cos\theta \to \theta = \cos^{-1}\left(\frac{L^2/4}{L^2/4+h^2}\right)$$

so, $S.A. = 4\int_{\pi/2}^{\cos^{-1}\left(\frac{L^2/4}{L^2/4+h^2}\right)} \frac{h}{\frac{h^2}{1-\cos\theta}}\frac{1}{\sqrt{h^2\frac{1+\cos\theta}{1-\cos\theta}}}\frac{-\sin\theta}{(1-\cos\theta)^2}h^2 d\theta$

$$S.A. = 4\int_{\pi/2}^{\cos^{-1}\left(\frac{L^2/4}{L^2/4+h^2}\right)} -d\theta = -4\theta\Big|_{\pi/2}^{\cos^{-1}\left(\frac{L^2/4}{L^2/4+h^2}\right)}$$

$$S.A. = 2\pi - 4\cos^{-1}\left(\frac{L^2}{L^2+4h^2}\right) = 4\left\{\frac{\pi}{2}-\cos^{-1}\left(\frac{L^2}{L^2+4h^2}\right)\right\}$$

$$\boxed{S.A. = 4\sin^{-1}\left(\frac{L^2}{L^2+4h^2}\right)}$$