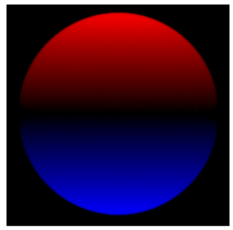
Electromagnetism

Electric dipole

Problem 1.- Calculate the dipole moment of a spherical surface of radius R that has charge density given by the following function in spherical coordinates:

$$\sigma = \sigma_0 \cos\theta$$



Solution: By definition, the dipole moment is the following integral.

$$\vec{p} = \int \vec{r}dq$$

We notice that due to symmetry, there is only dipole in the z-direction, so the integral can be simplified to:

$$\vec{p} = \hat{z}\int zdq = \hat{z}\int R\cos\theta dq$$

We consider differentials of charge given by differentials of area times the surface density.

$$dq = \sigma dA = \sigma_0 \cos\theta 2\pi R\sin\theta R d\theta = 2\pi R^2 \sigma_0 \sin\theta\cos\theta d\theta$$

Putting them together we get:

$$\vec{p} = \hat{z}\int_0^{\pi} R\cos\theta 2\pi R^2 \sigma_0 \sin\theta\cos\theta d\theta = 2\pi\sigma_0 R^3 \hat{z}\int_0^{\pi} \cos\theta\sin\theta\cos\theta d\theta$$

So, the dipole is:

$$\vec{p} = \frac{4}{3}\pi\sigma_0 R^3 \hat{z}$$

Problem 2.- Calculate the electric energy due to the electric field of a dipole moment p located at the origin of coordinates for the space outside a sphere of radius R centered at the dipole.

Solution: The electric field due to a dipole in spherical coordinates is given by:

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}, \quad E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}, \quad \text{and } E_\phi = 0$$

The energy density due to the electric field is $u = \frac{dU}{d\tau} = \frac{\varepsilon_0}{2} E^2$, so we need to integrate this density over the space outside the radius *R*:

$$U = \frac{\varepsilon_0}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} E^2 r^2 \sin\theta dr d\theta d\phi = \frac{\varepsilon_0}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \left(E_r{}^2 + E_\theta{}^2 + E_\phi{}^2 \right) r^2 \sin\theta dr d\theta d\phi$$

Integrating, we get:

$$U = \frac{\varepsilon_0}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{1}{16\pi^2 \varepsilon_0{}^2} \frac{4p^2 \cos^2\theta}{r^6} + \frac{1}{16\pi^2 \varepsilon_0{}^2} \frac{p^2 \sin^2\theta}{r^6} \right) r^2 \sin\theta dr d\theta d\phi$$

$$U = \frac{p^2}{16\pi\varepsilon_0} \int_R^\infty \int_0^\pi \left(3\cos^2\theta + 1 \right) \sin\theta r^{-4} dr d\theta$$

$$U = \frac{p^2}{4\pi\varepsilon_0} \int_R^\infty r^{-4} dr$$

$$U = \frac{p^2}{12\pi\varepsilon_0 R^3}$$

We notice that if *R* went to zero, the energy would diverge.