# Electromagnetism

## Gauss's Law

**Problem 1.-** Find the electric potential everywhere produced by a spherical distribution of charge whose density is given by:

$$\rho = \begin{cases} a(R-r) & r < R \\ 0 & r \geq R \end{cases}$$

**Solution:** First, we find the electric field:

For $r > R$, outside the sphere

$$E = \frac{k}{r^2}\int_0^R a(R-r)4\pi r^2 dr = \frac{4\pi ka}{r^2}\int_0^R (R-r)r^2 dr = \frac{4\pi ka}{r^2}\left(\frac{R^4}{3} - \frac{R^4}{4}\right) = \frac{\pi kaR^4}{3r^2}$$

For $r < R$, inside the sphere

$$E = \frac{k}{r^2}\int_0^r a(R-r)4\pi r^2 dr = \frac{4\pi ka}{r^2}\int_0^r (R-r)r^2 dr = \frac{4\pi ka}{r^2}\left(\frac{Rr^3}{3} - \frac{r^4}{4}\right) = 4\pi ka\left(\frac{Rr}{3} - \frac{r^2}{4}\right)$$

To find the potential we integrate:

For $r > R$, outside the sphere $V = -\int_\infty^r \frac{\pi kaR^4}{3r^2} dr = \frac{\pi kaR^4}{3r}$

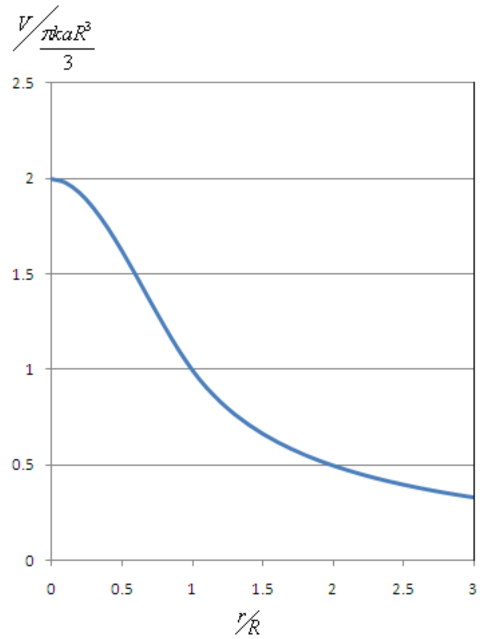
Notice the value of the potential at the boundary is: $V(R) = \frac{\pi kaR^3}{3}$

For $r < R$, inside the sphere

$$V = V(R) - \int_R^r 4\pi ka\left(\frac{Rr}{3} - \frac{r^2}{4}\right) dr = V(R) + 4\pi ka\left(\frac{Rr^2}{6} - \frac{r^3}{12}\right)\Bigg|_r^R$$

$$V = \frac{\pi kaR^3}{3} + 4\pi ka\left(\frac{R^3}{6} - \frac{R^3}{12}\right) - 4\pi ka\left(\frac{Rr^2}{6} - \frac{r^3}{12}\right)$$

$$V = \frac{\pi ka}{3}\left[2R^3 - 2Rr^2 + r^3\right]$$




**Problem 2.-** A sphere of radius $R$ has a volume distribution of charge given by $\rho = Cr^3$, where $C$ is a constant and $r$ is the distance to the center of the sphere. Calculate the magnitude of the electric field at $r = R/2$.

**Solution:** According to Gauss's theorem: $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}$, for a point located at $r = R/2$ the left side of the equation is: $\oint \vec{E} \cdot d\vec{a} = 4\pi \left(\frac{R}{2}\right)^2 E = \pi R^2 E$

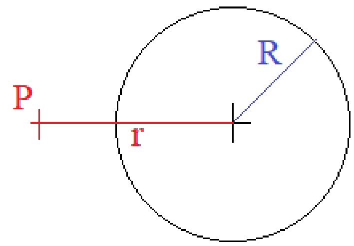
The charge enclosed can be calculated by integration. Notice that you cannot just multiply the density times the volume because the density is not constant.

$$Q_{enclosed} = \int_0^{R/2} \rho dV = \int_0^{R/2} Cr^3 (4\pi r^2 dr) = 4\pi C \int_0^{R/2} r^5 dr = \frac{4\pi C\left(\frac{R}{2}\right)^6}{6} = \frac{\pi C R^6}{96}$$

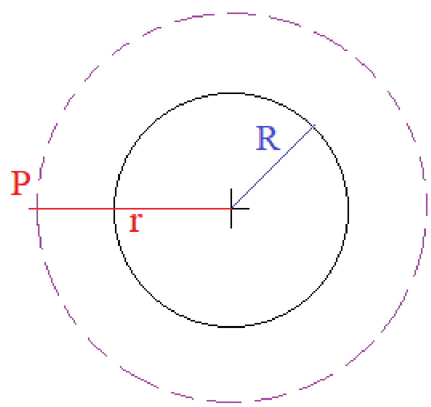
And using this result in the equation above we get: $\pi R^2 E = \frac{\pi C R^6}{96\varepsilon_0} \rightarrow E = \frac{CR^4}{96\varepsilon_0}$

**Problem 3.-** Calculate the electric field at a point P located at a distance r from the axis of a long cylindrical and homogeneous distribution of charge of density $\rho$ and radius R.

**Solution:** One possibility is that the point is outside the cylinder, like in this configuration:




Consider an enclosing surface with the shape of a cylindrical prism of radius r and height h.




According to Gauss's theorem: $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}$, so in this case:

$$E 2\pi r h = \frac{\rho h \pi R^2}{\varepsilon_0}$$

After simplification we get:

$$E = \frac{\rho R^2}{2\varepsilon_0 r} \qquad \text{for r > R}$$

Notice that the electric field drops as 1/r as opposed to the usual $1/r^2$ of point charges.
The other possibility is that r < R (P inside the cylinder) in which case the equation is:

$$E 2\pi r h = \frac{\rho h \pi r^2}{\varepsilon_0}$$

And after simplification:

$$E = \frac{\rho r}{2\varepsilon_0} \qquad \text{for r < R}$$

In this case the electric field is linearly proportional to the distance to the center, which is the same for a sphere uniformly charged.