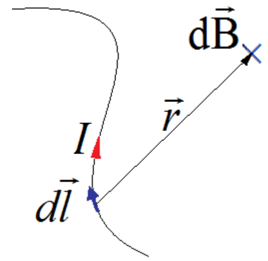
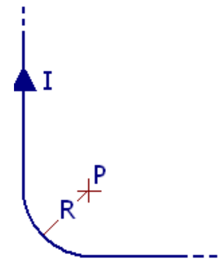
Electromagnetism

Biot-Savart law



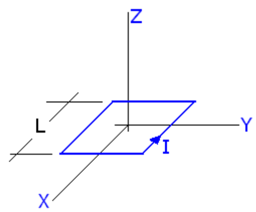
$$d\vec{B} = \frac{\mu_0 I d\vec{l} \times \vec{r}}{4\pi r^3}$$

Problem 1.- Calculate the magnetic field produced at point *P* due to the current carrying wire shown in the figure.



Solution: The two semi-infinite wires contribute together the same magnetic field that you get from just one infinite wire $B_1 = \frac{\mu_0 I}{2\pi R}$. The quarter of a circle contributes one fourth of a complete circular loop $B_2 = \frac{1}{4}\left(\frac{\mu_0 I}{2R}\right)$ and these vectors point in the same direction (towards the paper) so we just need to add their values: $B = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{8R}$

Problem 2.- An electric circuit has the shape of a square of side $L = 0.25$m and carries a current $I = 4$A as shown in the figure. Calculate the magnetic field at point $P = (4\text{m}, 5\text{m}, 3\text{m})$.



Solution: Using a numerical calculation we find:

$$\vec{B} = \left(5.12\times10^{-11}, 6.35\times10^{-11}, -3.25\times10^{-11}\right)\text{T}$$

Problem 3.- In the problem above calculate the magnetic field using the approximation of a dipole moment. That is, calculate the magnetic moment and then use the equation:

$\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{1}{r^3}\left(\dfrac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right)$ to find the magnetic field.

Solution: The magnetic moment is:

$\vec{m} = I\vec{a} = (0, 0, 0.25)$ where a is the area of the circuit.

We also know that $\vec{r} = (4, 5, 3)\text{m}$, which means $r = \sqrt{4^2 + 5^2 + 3^2}\,\text{m} = \sqrt{50}\text{m}$

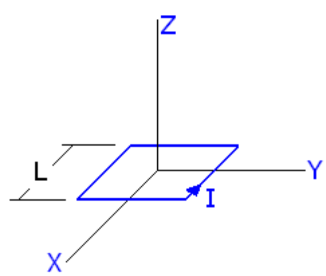
$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{\sqrt{50}^3}\left(\frac{3((0,0,0.25)\cdot(4,5,3))(4,5,3)}{\sqrt{50}^2} - (0,0,0.25)\right)$$

$$\vec{B} = \frac{10^{-7}}{\sqrt{50}^3}\left(\frac{2.25(4,5,3)}{50} - (0,0,0.25)\right) = \frac{10^{-7}}{\sqrt{50}^3}\left(0.18, 0.225, -0.115\right)\text{T}$$

$$\vec{B} = \left(5.09\times10^{-11}, 6.36\times10^{-11}, -3.3\times10^{-11}\right)\text{T}$$

This is very close to the numerically calculated vector.

Problem 3a.- An electric circuit has the shape of a square of side L = 0.1m and carries a current I=10A as shown in the figure. Calculate the magnetic field at point P = (3m, 4m, 0)
Use the approximation of a dipole moment.

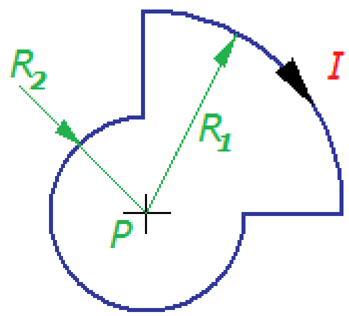


Solution: The magnetic moment is $\vec{m} = I\vec{a} = (0,0,0.1)$, vector $\vec{r}$ is $\vec{r} = (3,4,0)$ and $|\vec{r}| = 5$, so the magnetic field is:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right) = \frac{\mu_0}{4\pi}\frac{1}{5^3}\left(\frac{3((0,0,0.1)\cdot(3,4,0))(3,4,0)}{5^2} - (0,0,0.1)\right)$$

$$\vec{B} = \frac{10^{-7}}{5^3}(0,0,-0.1) = (0,0,-80\text{pT})$$

Problem 4.- Calculate the magnetic field produced at point *P* due to the current carrying wire shown in the figure. $I = 25$A, $R_1 = 0.88$m and $R_2 = 0.44$m
Notice that the arc with radius R_1 is a quarter of a circle.

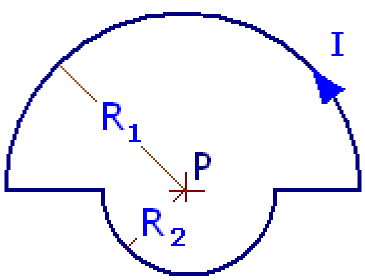


Solution: The straight wires do not contribute any magnetic field at point P. You can convince yourself of that by looking at the cross product $d\vec{\ell}\times\vec{r}$ in the Biot-Savart equation.

On the other hand, each arc contributes a magnetic field equal to its fraction of a complete circle. Then:

$$\vec{B} = \frac{1}{4}\frac{\mu_0 I}{2R_1} + \frac{3}{4}\frac{\mu_0 I}{2R_2} = \frac{\mu_0 I}{8}\left(\frac{1}{R_1} + \frac{3}{R_2}\right) = \frac{4\pi\times10^{-7}\times25}{8}\left(\frac{1}{0.88} + \frac{3}{0.44}\right) = \mathbf{31.2\mu T}$$

Problem 4a.- Calculate the magnetic field produced at point *P* due to the current carrying wire shown in the figure. $I = 15$A, $R_1 = 0.4$m and $R_2 = 0.2$m

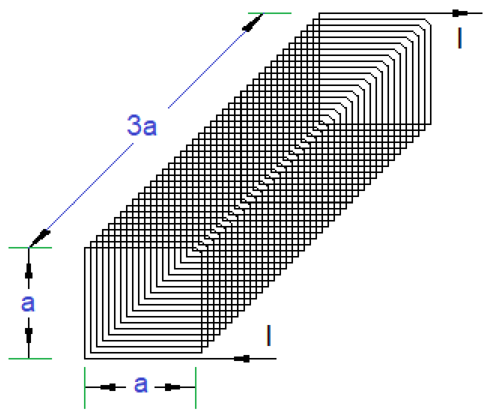


Solution: The straight wires do not contribute any magnetic field at point P. You can convince yourself of that by looking at the cross product $d\vec{\ell}\times\vec{r}$ in the Biot and Savart equation.

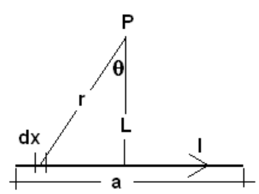
On the other hand, each semicircle contributes a magnetic field equal to half of a complete circle. Then:

$$\vec{B} = \frac{1}{2}\frac{\mu_0 I}{2R_1} + \frac{1}{2}\frac{\mu_0 I}{2R_2} = \frac{\mu_0 I}{4}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{4\pi\times 10^{-7}\times 15}{4}\left(\frac{1}{0.4} + \frac{1}{0.2}\right) = \mathbf{35.3\mu T}$$

Problem 5.- A solenoid is built with a square section of side *a*, *n* turns (with *n* very large) and length 3*a*. Calculate the magnetic field produced on the center of the first square when a current *I* flows in the solenoid.

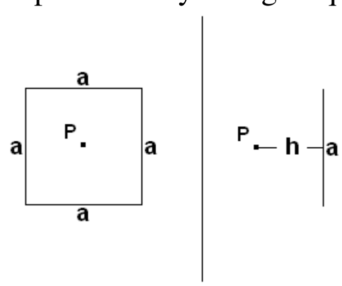


Solution: We first calculate the magnetic field produced by a piece of current carrying wire of length *a* at a distance *L* from its center.



$$B = \frac{\mu_0 I}{4\pi}\int\frac{dl\times\hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\int\frac{\cos\theta dx}{r^2} = \frac{\mu_0 I}{4\pi}\int\frac{\cos\theta d(L\tan\theta)}{(L/\cos\theta)^2} = \frac{\mu_0 aI}{2\pi L\sqrt{4L^2 + a^2}}$$

Then we calculate the magnetic field produced by a single square at a height *h* from its center.



We use the previous equation for the four segments that make the square. Each one contributes a field intensity equal to:

$$B_{segment} = \frac{\mu_o aI}{\pi\sqrt{4h^2 + a^2}\sqrt{4h^2 + 2a^2}}$$

However, these vectors do not point in the same direction. We need to add only the components that are normal to the square, so the sum of the four segments will be:

$$B_{square} = 4\frac{\mu_0 a I}{\pi\sqrt{4h^2+a^2}\sqrt{4h^2+2a^2}}\frac{a}{\sqrt{4h^2+a^2}} = \frac{4\mu_0 a^2 I}{\pi(4h^2+a^2)\sqrt{4h^2+2a^2}}$$

Next, we sum over all the squares in the solenoid. Since n is large, we can approximate the sum with an integral.

$$B = \frac{n}{3a}\int_0^{3a}\frac{4\mu_0 a^2 I}{\pi(4h^2+a^2)\sqrt{4h^2+2a^2}}\,dh$$

We change variables to $h = xa$

$$B = \frac{4\mu_0 n I}{3\pi a}\int_0^{3}\frac{dx}{(4x^2+1)\sqrt{4x^2+2}}$$

Integrating numerically, we get

$$B = \frac{4\mu_0 n I}{3\pi a}(0.3859)$$

You can confirm that this result is within 2% of the value of a half-infinite solenoid.