Electromagnetism

Math background

Problem 1.- Find the gradient of the following functions:

a) $\phi = \dfrac{1}{x^2+y^2+z^2} + \dfrac{1}{(x-2)^2+y^2+z^2}$

b) $\phi = x + y + z$

Solution:

a) $\nabla\phi = \nabla\left(\dfrac{1}{x^2+y^2+z^2} + \dfrac{1}{(x-2)^2+y^2+z^2}\right) = \dfrac{-2x}{\left(x^2+y^2+z^2\right)^2}$

$$\nabla\phi = \left[\frac{-2x}{\left(x^2+y^2+z^2\right)^2} + \frac{-2(x-2)}{\left((x-2)^2+y^2+z^2\right)^2}\right]\hat{x} + \left[\frac{-2y}{\left(x^2+y^2+z^2\right)^2} + \frac{-2y}{\left((x-2)^2+y^2+z^2\right)^2}\right]\hat{y}$$

$$+\left[\frac{-2z}{\left(x^2+y^2+z^2\right)^2} + \frac{-2z}{\left((x-2)^2+y^2+z^2\right)^2}\right]$$

An alternative way of writing this result is using the change of variable:

$\vec{r} = (x, y, z)$ and $\vec{r}' = (x-2, y, z)$

So:

$$\phi = \frac{1}{r^2} + \frac{1}{r'^2}$$

and the gradient is:

$$\nabla\phi = \frac{-2\vec{r}}{r^4} + \frac{-2\vec{r}'}{r'^4}$$

b) $\nabla\phi = \nabla(x+y+z) = (1,1,1)$

Problem 2.- Find the gradient of the following vectors:

a) $\vec{v} = (xy, yz, zx)$

b) $\vec{v} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, 0\right)$, if this vector represents an electric field, can you say anything about where the charge is located?

Solution:

a) $\nabla \cdot \vec{v} = y + z + x$

b) $\nabla \cdot \vec{v} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}$

$$\nabla \cdot \vec{v} = \frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2} + \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} = 0$$

Since the divergence of this vector is zero, if this vector represented an electric field, the density of charge would be zero everywhere. However, notice that the divergence cannot be calculated on the z-axis because the denominator is zero there. We can then conclude that the charge that produces the electric field is at x=0, y=0.

Problem 3.- Find the curl of the following vectors:

a) $\vec{v} = (-y, x, z)$

b) $\vec{v} = \left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$

Solution:

a) $\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 2\hat{z}$

b) $\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix} =$

$$\nabla \times \vec{v} = \hat{x}\left[\frac{\partial}{\partial y}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right]$$

$$-\hat{y}\left[\frac{\partial}{\partial x}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right]$$

$$+\hat{z}\left[\frac{\partial}{\partial x}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial y}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right]$$

$$\nabla\times\vec{v} = \hat{x}\left[\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{-3zy}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$-\hat{y}\left[\frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$+\hat{z}\left[\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$\nabla\times\vec{v} = (0,0,0)$$

Problem 4.- Prove the following identity:

$$\nabla\times\left(\nabla\times\vec{v}\right) \equiv \nabla\left(\nabla\cdot\vec{v}\right) - \nabla^2\vec{v}$$

Solution: First, we calculate the left side of the equation:

$$\nabla\times\left(\nabla\times\vec{v}\right) = \nabla\times\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, -\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla\times\left(\nabla\times\vec{v}\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} & -\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{vmatrix}$$

$$\nabla\times\left(\nabla\times\vec{v}\right) =$$

$$\hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right] +$$

$$\hat{z}\left[\frac{\partial}{\partial x}\left(-\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]$$

$$\nabla\times\left(\nabla\times\vec{v}\right) =$$

$$\hat{x}\left[\frac{\partial^2 v_y}{\partial x\partial y} + \frac{\partial^2 v_z}{\partial x\partial z} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2}\right] + \hat{y}\left[\frac{\partial^2 v_z}{\partial y\partial z} + \frac{\partial^2 v_x}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x^2} - \frac{\partial^2 v_y}{\partial z^2}\right] +$$

$$\hat{z}\left[\frac{\partial^2 v_x}{\partial x\partial z} + \frac{\partial^2 v_y}{\partial y\partial z} - \frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2}\right]$$

Now we look at the right side of the equation:

$$\nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}$$

$$\nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}=\nabla\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(v_x,v_y,v_z)$$

$$=\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z},\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial y\partial z},\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}+\frac{\partial^2 v_z}{\partial z^2}\right)-$$

$$\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2},\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2},\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

$$=\left(\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}-\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial^2 v_x}{\partial z^2},\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_z}{\partial y\partial z}-\frac{\partial^2 v_y}{\partial x^2}-\frac{\partial^2 v_y}{\partial z^2},\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}-\frac{\partial^2 v_z}{\partial x^2}-\frac{\partial^2 v_z}{\partial y^2}\right)$$

They are the same!