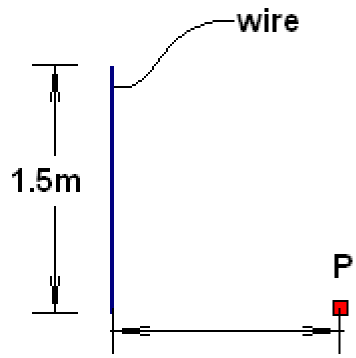
Electromagnetism

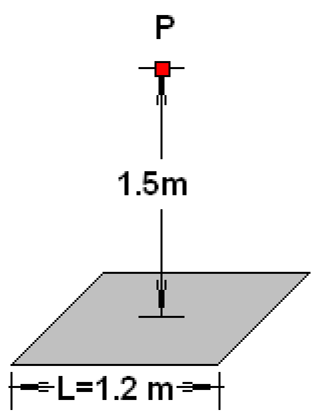
Electric field

Problem 1.-

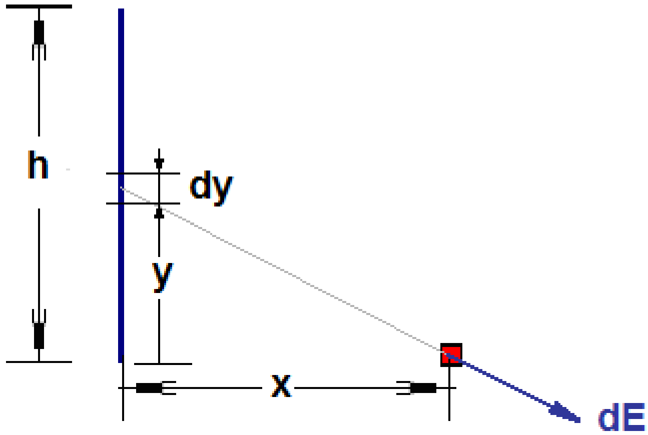
a) Find the electric field at point P produced by the wire shown in the figure. Consider that the wire has a uniform linear charge distribution of $\lambda = 2.25 \mu C / m$



b) Find the electric field at point P produced by the square of uniform surface charge distribution $\sigma = 3.4 \mu C / m^2$



Solution:
Part a)



We consider the electric field produced by a differential of wire as shown in the figure. The differential of charge is $dq = \lambda dy$ and the distance to point P is $d = \sqrt{x^2 + y^2}$, so the magnitude of the electric field is:

$$dE = \frac{k\lambda dy}{x^2 + y^2}$$

The two components can be found by multiplying by the cosine and sine of the angle with the horizontal:

$$dE_x = \frac{k\lambda dy}{x^2 + y^2}\frac{x}{\sqrt{x^2 + y^2}} \text{ and } dE_y = -\frac{k\lambda dy}{x^2 + y^2}\frac{y}{\sqrt{x^2 + y^2}}$$

Now, all we need to do is integrate the expressions above to find the electric field:

For the x-component the integral is: $E_x = \int_0^h \frac{k\lambda dy}{x^2 + y^2}\frac{x}{\sqrt{x^2 + y^2}} = k\lambda x \int_0^h \frac{dy}{\left(x^2 + y^2\right)^{3/2}}$

Change of variable

$$y = x\tan\theta \rightarrow \left(x^2 + y^2\right)^{3/2} = x^3\sec^3\theta, dy = x\sec^2\theta d\theta, y = 0 \rightarrow \theta = 0, y = h \rightarrow \theta = \tan^{-1}\left(\frac{h}{x}\right)$$

$$E_x = k\lambda x \int_0^{\tan^{-1}(h/x)} \frac{x\sec^2\theta d\theta}{x^3\sec^3\theta} = \frac{k\lambda}{x}\int_0^{\tan^{-1}(h/x)} \cos\theta d\theta = \frac{k\lambda}{x}\sin\theta\Big|_0^{\tan^{-1}(y/x)}$$

$$E_x = \frac{k\lambda h}{x\sqrt{x^2 + h^2}}$$

For the y-component the integral is:

$$E_y = \int_0^h -\frac{k\lambda dy}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} = -k\lambda\int_0^h \frac{ydy}{\left(x^2+y^2\right)^{3/2}} = k\lambda\frac{1}{\sqrt{x^2+y^2}}\Bigg|_0^h$$

$$= -k\lambda\frac{1}{x} + k\lambda\frac{1}{\sqrt{x^2+h^2}}$$

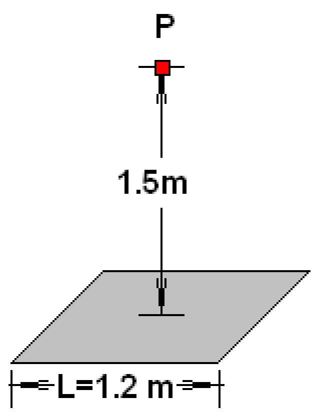
$$E_y = \frac{k\lambda}{x}\left(\frac{x}{\sqrt{x^2+h^2}} - 1\right)$$

With the values of the problem:

$$E_x = \frac{9\times10^9\times2.25\times10^{-6}\times1.5}{0.8\times\sqrt{1.5^2+0.8^2}} = \textbf{22,300 V/m}$$

$$E_y = \frac{9\times10^9\times2.25\times10^{-6}}{0.8}\left(\frac{0.8}{\sqrt{1.5^2+0.8^2}} - 1\right) = \textbf{-13,400 V/m}$$

Part b)



If you consider the contribution of a differential of area of the square, the electric field will be:

$$dE = \frac{k\sigma dxdy}{x^2+y^2+h^2}$$

Then, to find the z-component we multiply by the cosine of the angle between the z-axis and the electric field vector:

$$dE_z = \frac{k\sigma dxdy}{x^2+y^2+h^2}\frac{h}{\sqrt{x^2+y^2+h^2}}$$

You should realize that the expression $\frac{dxdy}{x^2+y^2+h^2}\frac{h}{\sqrt{x^2+y^2+h^2}}$ is the differential of solid angle produced at point P by $dxdy$, so the electric field will be:

$$E_z = k\sigma \times S.A.$$

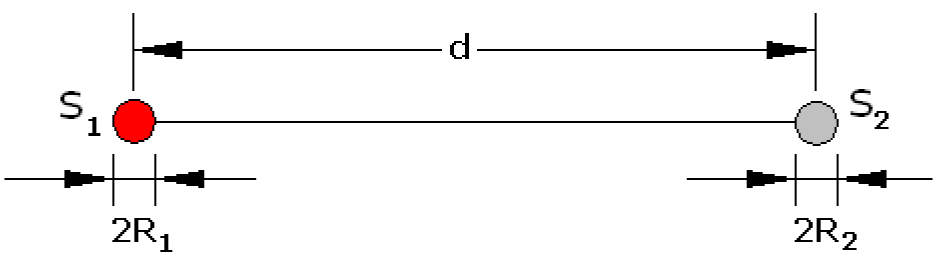
Where $S.A.$ is the solid angle produced by the square. You can use the expression derived elsewhere: $S.A. = 2\pi - 4\cos^{-1}\left(\frac{L^2/4}{h^2+L^2/4}\right)$ to calculate the electric field:

$$E_z = k\sigma\left[2\pi - 4\cos^{-1}\left(\frac{L^2/4}{h^2+L^2/4}\right)\right]$$

With the values of the problem:

$$E_z = 9\times10^9 \times 3.4\times10^{-6}\left[2\pi - 4\cos^{-1}\left(\frac{1.2^2/4}{1.5^2+1.2^2/4}\right)\right] = \mathbf{16{,}900\ V/m}$$

Problem 2.- Two identical conducting spheres S_1 and S_2 have radii $R_1= R_2=0.01$m and are separated by a long distance d=1m. Initially one sphere is charged with $Q_1=2\mu C$ and the other is neutral ($Q_2=0$). Then you bring a third identical conducting sphere (initially neutral) and make contact with the charged sphere first and then with the neutral one. After that, you take the third sphere very far away. What is now the force between S_1 and S_2?



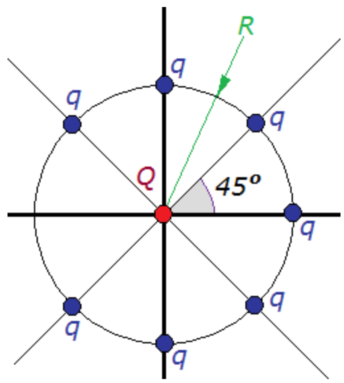
Solution: When two identical spheres touch they will exchange charge until the potential is the same on both spheres, so until the charge is shared by in equal quantities.

After the third sphere touches S_1, the charge on S_1 will be $Q_1=1\mu C$ (the other $1\mu C$ will be in the third sphere).

After the third sphere touches S_2, the charge on S_2 will be $Q_2=1/2\mu C$

The force is: $F = \frac{1}{4\pi\varepsilon_\circ}\frac{Q_1Q_2}{d^2} = 9\times10^9\frac{(1\times10^{-6})(0.5\times10^{-6})}{1^2} = 4.5\times10^{-3}N$

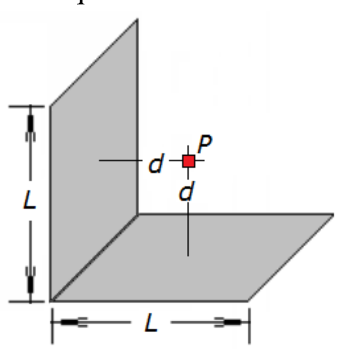
Problem 3.- Find the force on the positive charge Q=5μC at the origin of coordinates due to the seven negative charges q=-1μC located around the circle of radius R=0.45m at the positions shown in the figure.



Solution: Notice that all the forces cancel each other except for the one due to the charge on the positive x-axis. The force is:

$$F = \frac{1}{4\pi\varepsilon_{\circ}}\frac{Qq}{d^2} = 9\times10^9\,\frac{(5\times10^{-6})(1\times10^{-6})}{0.45^2} = 0.222N$$

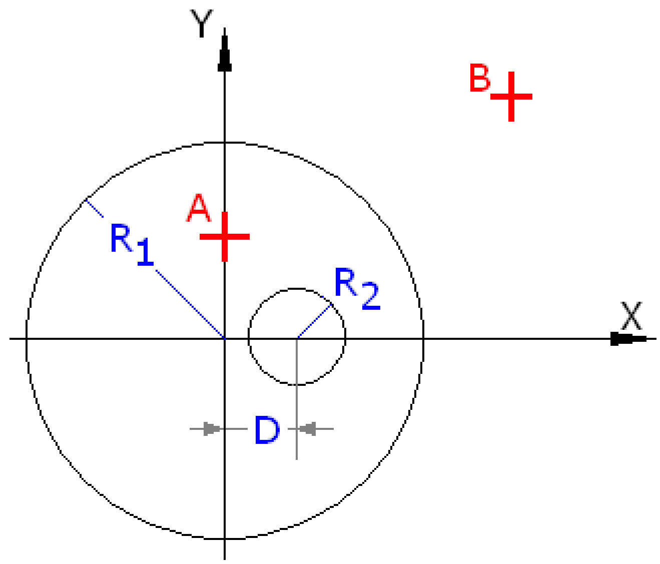
Problem 4.- Find the electric field at point P, which is at a distance d=1.2m from the center of two squares of side L=2.4m and uniform surface charge density $\sigma = 3.4\mu C/m^2$



Solution: The electric field produced by each square is: $E_1 = E_2 = k\sigma\frac{4\pi}{6}$ =64,100 V/m

Adding the two vectors we get: $E = \sqrt{E_1{}^2 + E_2{}^2}$ =**90,600 V/m**

Problem 5.- A sphere of radius R_1=1m is located with its center at the origin of coordinates and has a charge of Q_1=4nC uniformly distributed over its surface. Another sphere of radius R_2=0.3m has its center on the x-axis at a distance of D=0.4m from the origin of coordinates and has a charge of Q_2=-3nC also uniformly distributed over its surface.
Calculate the electric field at point B=(1.5m,1.5m,0)



Solution: Two vectors contribute to the electric field. The large sphere contributes:

$$E_1 = \left|\frac{kQ_1}{1.5^2 + 1.5^2}\right| = 8\text{V/m}$$

And the smaller sphere

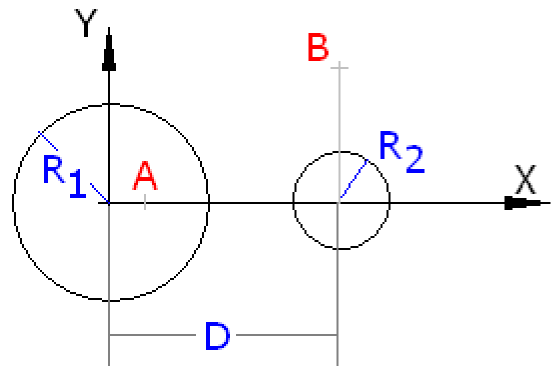
$$E_2 = \left|\frac{kQ_2}{1.1^2 + 1.5^2}\right| = 7.8\text{V/m}$$

Adding the vectors component by component, we get:

$$\vec{E}_1 = (8\cos 45^\circ, 8\sin 45^\circ)$$
$$\vec{E}_2 = (-7.8\cos 53.7^\circ, -7.8\sin 53.7^\circ)$$
$$\vec{E} = \textbf{(1.04, -0.636) V/m}$$

Problem 6.- A sphere of radius R_1=12cm is located with the center at the origin of coordinates and has a charge of 4nC uniformly distributed over its surface. Another sphere of radius R_2=6cm has its center on the x-axis at a distance of D=25cm from the origin of coordinates and has a charge of -3nC also uniformly distributed over its surface.

Calculate the electric field at point A=(4,0,0)



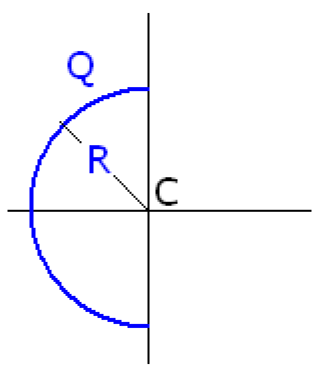
Solution:

The electric field at point A=(4,0,0): Notice that the contribution to the electric field from the charge of the large sphere is zero. This is because the charge is uniformly distributed on its surface and point A is *inside* the sphere.
On the other hand, the smaller sphere does produce an electric field at point A. Since the charge distribution is spherical and point A is outside the smaller sphere, its electric field is the same as if all the charge were located at its center, so:

$$E = k\frac{q}{d^2} = 9\times10^9 \frac{3\times10^{-9}}{(0.21)^2} = \mathbf{612\ V/m}$$

Notice that in the calculation of electric field we ignored the sign of the charge. What the sign indicates is whether the vector is toward the charge or away from it. In this case the electric field vector is directed in the positive x-direction.

Problem 7.- Calculate the electric field at point "C" due to the uniformly distributed charge Q on the semicircle of radius R.



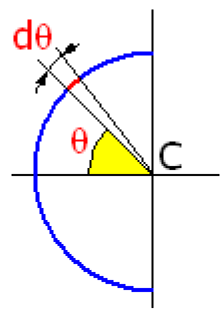
Solution: In problems where there is a continuous distribution of charges the standard procedure to find electric field or potential is:

- Divide the continuous distribution in small pieces (differentials) that can be treated as point charges.
- Calculate the field or potential produced by the differential. Be careful that in the case of electric field you will need to calculate the components independently.
- All distances, charges and angles should be written in terms of the variables chosen.
- Integrate over those variables.

The problem above falls precisely in the category that we just mentioned. So let us divide the charge in small pieces. To do this, notice that the charge is uniformly distributed, so the linear density of charge is:

$$\lambda = \frac{\text{charge}}{\text{length}} = \frac{Q}{\pi R}$$

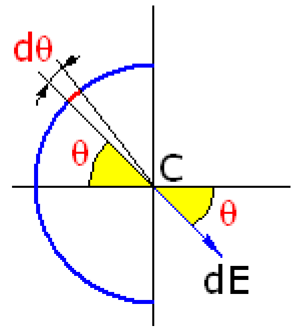
Then, we divide the arc in differentials as shown in the figure:



The length of the differential of arc is $d\ell = R d\theta$ and the differential of charge is $dq = \lambda d\ell = \lambda R d\theta = \frac{Q}{\pi R} R d\theta = \frac{Q}{\pi} d\theta$

We first calculate the magnitude of the electric field produced by the differential of charge, which is as if it were a point: $dE = \frac{kdq}{d^2} = \frac{k\frac{Q}{\pi}d\theta}{R^2} = \frac{kQd\theta}{\pi R^2}$

But this is the magnitude of the vector. It has an x-component and a y-component. By the symmetry of the problem, we notice that we only need to calculate the x-component.



The x-component of the electric field is:

$$dE_x = dE\cos\theta = \frac{kQd\theta}{\pi R^2}\cos\theta$$

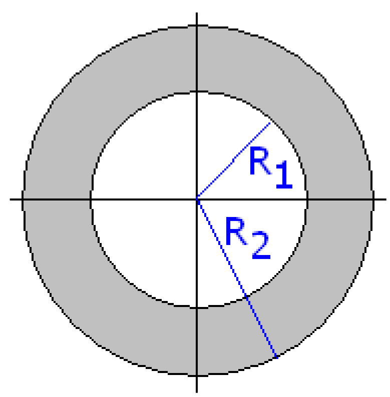
Now we need to integrate this expression to find the electric field:

$$E_x = \int dE_x = \int_{-\pi/2}^{\pi/2} \frac{kQd\theta}{\pi R^2}\cos\theta = \frac{2kQ}{\pi R^2}$$

Problem 8.- A spherical shell has internal radius R_1 and external radius R_2 and contains a uniform distribution of charge with density ρ

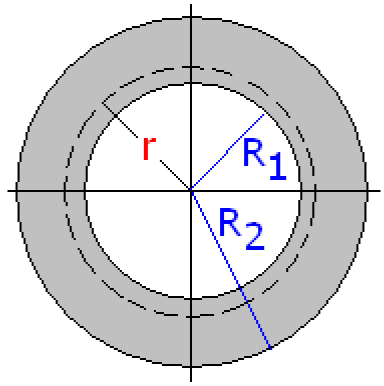
Calculate the electric field for a point at a distance "r" from the center of the shell. Consider 3 cases:

a) $r<R_1$
b) $R_1<r<R_2$
c) $r>R_2$



Solution: Inside the shell, when $r<R_1$ the electric field is zero because if you consider a spherical surface and apply Gauss' law, the charge enclosed is zero.

At a point in the shell, where $R_1<r<R_2$ we can use Gauss's law, but the charge enclosed will only be the one in the shell between R_1 and r.



The charge can be calculated by multiplying the volume of the shell (between R_1 and r) by the density of charge. This works because the density is constant, but if it were a function of the radius, we would have to divide the shell in layers (like an onion) and integrate layer by layer.

The charge enclosed is: $Q_{enclosed} = \rho\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R_1^3\right)$

Using Gauss's law: $E = \frac{kQ_{enclosed}}{r^2} = \frac{k\rho\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R_1^3\right)}{r^2} = \frac{4}{3}\pi k\rho\left(r - R_1^3 / r^2\right)$

Finally, outside the sphere you can calculate the electric field as if all the charge were located at the center of the shell. The total charge is:

$Q = \rho\left(\frac{4}{3}\pi R_2^3 - \frac{4}{3}\pi R_1^3\right)$

The electric field is $E = \frac{kQ}{r^2} = \frac{4}{3}\pi\frac{k\rho\left(R_2^3 - R_1^3\right)}{r^2}$