Electromagnetism

Gauss law

Problem 1.- Find the electric potential everywhere produced by a spherical distribution of charge whose density is given by:
$\rho = a(R-r) \quad r < R$

Solution: First, we find the **electric field:**

For $r > R$, **outside the sphere**

$$E = \frac{k}{r^2}\int_0^R a(R-r)4\pi r^2 dr = \frac{4\pi ka}{r^2}\int_0^R (R-r)r^2 dr = \frac{4\pi ka}{r^2}\left(\frac{R^4}{3}-\frac{R^4}{4}\right) = \frac{\pi kaR^4}{3r^2}$$

For $r < R$, **inside the sphere**

$$E = \frac{k}{r^2}\int_0^r a(R-r)4\pi r^2 dr = \frac{4\pi ka}{r^2}\int_0^r (R-r)r^2 dr = \frac{4\pi ka}{r^2}\left(\frac{Rr^3}{3}-\frac{r^4}{4}\right) = 4\pi ka\left(\frac{Rr}{3}-\frac{r^2}{4}\right)$$

Now to find the **potential** we integrate:

For $r > R$, **outside the sphere** $\quad V = -\int_\infty^r \frac{\pi kaR^4}{3r^2} dr = \frac{\pi kaR^4}{3r}$

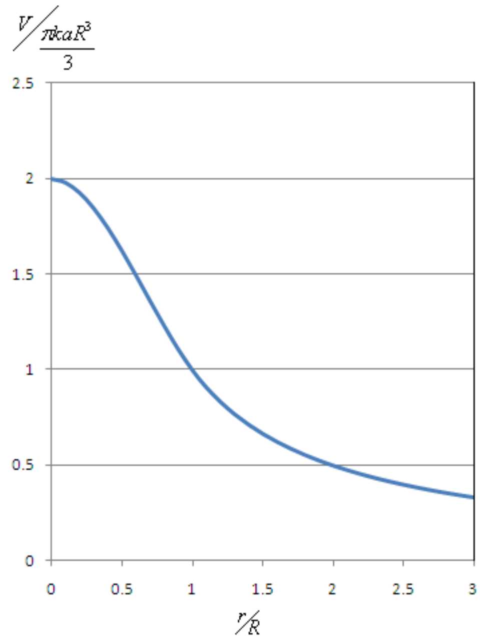
Notice the value of the potential at the boundary is: $V(R) = \frac{\pi kaR^3}{3}$

For $r < R$, **inside the sphere**

$$V = V(R) - \int_R^r 4\pi ka\left(\frac{Rr}{3}-\frac{r^2}{4}\right)dr = V(R) + 4\pi ka\left(\frac{Rr^2}{6}-\frac{r^3}{12}\right)\Bigg|_r^R$$

$$V = \frac{\pi kaR^3}{3} + 4\pi ka\left(\frac{R^3}{6}-\frac{R^3}{12}\right) - 4\pi ka\left(\frac{Rr^2}{6}-\frac{r^3}{12}\right)$$

$$V = \frac{\pi ka}{3}\left[2R^3 - 2Rr^2 + r^3\right]$$



Problem 2.- A sphere of radius R has a volume distribution of charge given by $\rho = Cr^3$, where C is a constant and r is the distance to the center of the sphere. Calculate the magnitude of the electric field at $r=R/2$.

Solution: According to Gauss's theorem: $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_\circ}$, for a point located at $r=R/2$ the left side of the equation is: $\oint \vec{E} \cdot d\vec{a} = 4\pi\left(\frac{R}{2}\right)^2 E = \pi R^2 E$

The charge enclosed can be calculated by integration. Notice that you cannot just multiply the density times the volume because the density is not constant.

$$Q_{enclosed} = \int_0^{R/2} \rho dV = \int_0^{R/2} Cr^3(4\pi r^2 dr) = 4\pi C \int_0^{R/2} r^5 dr = \frac{4\pi C\left(\frac{R}{2}\right)^6}{6} = \frac{\pi CR^6}{96}$$

And using this result in the equation above we get: $\pi R^2 E = \frac{\pi CR^6}{96\varepsilon_\circ} \rightarrow E = \frac{CR^4}{96\varepsilon_\circ}$