Electromagnetism

Poisson equation

Problem 1.- An electric potential is described by the equation:

$$V = \frac{q\cos 2\theta}{4\pi\varepsilon_\circ r}$$

Calculate the density of charge using Poisson's equation. For the LaPlacian you can use the formulas of the spherical coordinates' version of ∇^2

Solution:

$$\rho = -\varepsilon_\circ \nabla^2 V = -\varepsilon_\circ \nabla^2 \frac{q\cos 2\theta}{4\pi\varepsilon_\circ r} = -\frac{q}{4\pi}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{\cos 2\theta}{r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\frac{\cos 2\theta}{r}\right)\right)$$

Notice that we do not need to calculate the azimuthal term because V is independent of ϕ

Taking the derivatives, the partials with respect to r give zero, but the angular part is:

$$\rho = -\frac{q}{4\pi}\left(\frac{1}{r^3 \sin\theta}\frac{\partial}{\partial\theta}(-2\sin\theta\sin 2\theta)\right) = \rho = -\frac{q}{4\pi}\left(\frac{1}{r^3 \sin\theta}\frac{\partial}{\partial\theta}(-2\sin\theta\sin 2\theta)\right) =$$

$$\rho = -\frac{q}{4\pi}\left(\frac{1}{r^3 \sin\theta}(-2\cos\theta\sin 2\theta - 4\sin\theta\cos 2\theta)\right) =$$

$$\rho = \frac{q}{\pi}\left(\frac{1}{r^3}(\cos^2\theta + \cos 2\theta)\right) = \frac{q}{\pi r^3}(3\cos^2\theta - 1)$$

Problem 2.- The time averaged potential of a hydrogen atom is:

$$V = \frac{q}{4\pi\varepsilon_\circ}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$$

Calculate the density of charge using Poisson's equation. For the LaPlacian you can use the formula in spherical coordinates of ∇^2

Solution: The LaPlacian in spherical coordinates is: $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \ldots$

When applied to the potential given in the problem we get:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left[\frac{q}{4\pi\varepsilon_\circ}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)\right]\right) = \frac{q}{4\pi\varepsilon_\circ}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left[e^{-\alpha r}\left(\frac{1}{r} + \frac{\alpha}{2}\right)\right]\right)$$

$$\nabla^2 V = \frac{q}{4\pi\varepsilon_\circ}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(-\alpha e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)+e^{-\alpha r}\left(-\frac{1}{r^2}\right)\right)=\frac{q}{4\pi\varepsilon_\circ}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-\alpha e^{-\alpha r}\left(r+\frac{\alpha r^2}{2}\right)-e^{-\alpha r}\right)$$

$$\nabla^2 V = \frac{q}{4\pi\varepsilon_\circ}\frac{1}{r^2}\left(\alpha^2 e^{-\alpha r}\left(r+\frac{\alpha r^2}{2}\right)-\alpha e^{-\alpha r}\left(1+\alpha r\right)+\alpha e^{-\alpha r}\right)=\frac{q}{4\pi\varepsilon_\circ}\frac{\alpha^3 e^{-\alpha r}}{2}$$

The Poisson equation gives: $\rho = -\varepsilon_\circ \nabla^2 V = -\frac{q\alpha^3 e^{-\alpha r}}{8\pi}$