Thermal Physics

Energy and temperature

Problem 1.- The degeneracy of an ideal gas at constant volume is

$$g = \mathrm{CU}^{3\mathrm{N}/2}$$

(a) Prove that the average energy of a particle in an ideal gas is $3/2k_BT$.

(b) What can you conclude from the fact that $\left(\frac{\partial^2\sigma}{\partial \mathrm{U}^2}\right)_\mathrm{N}$ is negative?

Suggestion: change variables to use temperature instead of entropy in the derivative.

Solution: We can write the equation of the definition of temperature as follows:

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial \mathrm{U}}\right)_\mathrm{N}$$

The problem already states what the degeneracy is, so we can get the entropy as a function of energy:

$$\sigma = \log(\mathrm{g}) = \log(\mathrm{CU}^{3\mathrm{N}/2}) = \log(\mathrm{C}) + 3\mathrm{N}/2\log(\mathrm{U})$$

After taking partial derivative:

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial \mathrm{U}}\right)_\mathrm{N} = \frac{3\mathrm{N}}{2\mathrm{U}} \rightarrow \mathrm{U} = \frac{3}{2}\mathrm{N}\tau$$

This result is well known from elemental studies of ideal gases (recall that $\tau = k_BT$).
Taking second derivative to the entropy:

$$\left(\frac{\partial^2\sigma}{\partial \mathrm{U}^2}\right)_\mathrm{N} = \left(\frac{\partial\left(\frac{3\mathrm{N}}{2\mathrm{U}}\right)}{\partial \mathrm{U}}\right)_\mathrm{N} = -\frac{3\mathrm{N}}{2\mathrm{U}^2}$$, which is always negative.

Is there any significance in the fact that the second derivative of the entropy with respect to the energy is always negative? Yes, this is apparent when you change the variable to temperature, instead of energy:

$$\left(\frac{\partial^2\sigma}{\partial \mathrm{U}^2}\right)_\mathrm{N} = \left(\frac{\partial\left(\frac{1}{\tau}\right)}{\partial \mathrm{U}}\right)_\mathrm{N} = -\frac{1}{\tau^2}\frac{\partial\tau}{\partial \mathrm{U}}$$

The fact that this term is negative means that the derivative of the temperature with respect to the energy is always positive, i.e. the more energy you add, the higher the temperature.