Thermal Physics

States of two non-interacting identical spinless particles

Problem 1.- Consider two spinless bosons confined to a cubic box of side L. Calculate the first 5 energy levels of the system and their multiplicities.

Calculate the average kinetic energy if the energy of the ground state is $6Eo$ and the temperature is $T = 10\frac{Eo}{k_B}$

Note 1: Assume the particles are non-interacting.
Note 2: The particles are identical, so be careful not to count multiple times levels that cannot be distinguished.

Solution: For one particle alone, the energy levels correspond to the following quantum numbers:

State	nx	ny	nz	Energy	g
a	1	1	1	3Eo	1
b1	1	1	2	6Eo	3
b2	1	2	1	6Eo	
b3	2	1	1	6Eo	
c1	1	2	2	9Eo	3
c2	2	1	2	9Eo	
c3	2	2	1	9Eo	
d1	1	1	3	11Eo	3
d2	1	3	1	11Eo	
d3	3	1	1	11Eo	
e	2	2	2	12Eo	1

States of two particles will be combinations of these states, for example:

aa	ba	ca	da	ea
ab	bb	cb	db	eb
ac	bc	cc	dc	ec
ad	bd	cd	dd	ed
ae	be	ce	de	ee

Notice, however that states like ab and ba are indistinguishable, so we should count them only once:

aa	ba	ca	da	ea
ab	bb	cb	db	eb
ac	bc	cc	dc	ec
ad	bd	cd	dd	ed
ae	be	ce	de	ee

Of the states that are left, the energies are the sum of the one-particle energies, as follows (in terms of E_o):

6				
9	12			
12	15	18		
14	17	20	22	
15	18	21	23	24

Let us analyze the levels in order of energy:

Lowest energy level: It has an energy of 6Eo and it happens for the unique combination aa, so its degeneracy is 1.

Second energy level: With an energy of 9Eo, one particle is in state a, while the other is in state b. Since a has multiplicity = 1 and b has multiplicity = 3, the combination will have multiplicity 3 (the product of 1×3).

Third energy level: The energy will be 12Eo and there are two ways that we can get this: By combining states a and c, which gives three possibilities and combining two identical states b. This latter case should be analyzed carefully: There are 9 combinations:

b1b1	b2b1	b3b1
b1b2	b2b2	b3b2
b1b3	b2b3	b3b3

But notice that 3 combinations are repeated, so only six are really different:

b1b1	b2b1	b3b1
b1b2	b2b2	b3b2
b1b3	b2b3	b3b3

So this third energy level has multiplicity equal to 3 (from the combination ac) + 6 (from the combination bb) = 9.

Fourth energy level: This state has energy of 14 Eo and it is formed by combining states a and d, which gives a multiplicity of 1×3=3

Fifth energy level: With an energy of 15Eo, this level can be formed by combining states a and e which has a multiplicity of 1 and states b and c with a multiplicity of 3×3=9, giving a total multiplicity of 1+9=10.
Notice that the nine combinations of states b and c are different, so we count all of them:

b1c1	b2c1	b3c1
b1c2	b2c2	b3c2
b1c3	b2c3	b3c3

Summarizing, for two identical spinless bosons:

Level	Energy	Multiplicity
1	6Eo	1
2	9Eo	3
3	12Eo	9
4	14Eo	3
5	15Eo	10

Now, to calculate the average kinetic energy we only consider these first 5 states in the partition function:

$$\langle K.E.\rangle = \frac{6E_o(1)e^{\frac{-6E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + 9E_o(3)e^{\frac{-9E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + 12E_o(9)e^{\frac{-10E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + 14E_o(3)e^{\frac{-12E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + 15E_o(10)e^{\frac{-15E_o}{k_B\left(10\frac{E_o}{k_B}\right)}}}{(1)e^{\frac{-6E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + (3)e^{\frac{-9E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + (9)e^{\frac{-10E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + (3)e^{\frac{-12E_o}{k_B\left(10\frac{E_o}{k_B}\right)}} + (10)e^{\frac{-15E_o}{k_B\left(10\frac{E_o}{k_B}\right)}}}$$

$$\langle K.E.\rangle = E_o\,\frac{6e^{-0.6} + 27e^{-0.9} + 108e^{-1} + 42e^{-1.2} + 150e^{-1.5}}{1e^{-0.6} + 3e^{-0.9} + 9e^{-1} + 3e^{-1.2} + 10e^{-1.5}} = 12.2E_o$$

The average energy is so high that this approximation does not seem correct in this case. More levels should be taken into account for an accurate calculation.