Thermal Physics

Thermal Noise in Circuits

Problem 1.- Find an expression for the root-mean-square voltage $\sqrt{\langle V^2 \rangle}$ of thermal noise in a capacitor C at temperature T. What is this in volts when T = 300K and C = 47pF?

Solution: The energy stored in a capacitor is given by: $\varepsilon_s = \frac{1}{2}\frac{(sq_e)^2}{C}$, where C is the capacitance, s is the number of electrons making up the charge (a negative value of s would indicate absence of electrons or positive charge).

The voltage in the capacitor is $V = -\frac{sq_e}{C}$ and we are interested in the average of the square of this value, which can be calculated with Boltzmann factors:

$$\langle V^2 \rangle = \frac{\sum_{s=-\infty}^{\infty} V^2 e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}}}{\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}}} = \frac{\sum_{s=-\infty}^{\infty} \left(\frac{sq_e}{C}\right)^2 e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}}}{\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}}}$$

Since the value of $\frac{q_e}{C}$ is very small we can convert the summations into integrals:

$$\langle V^2 \rangle = \frac{\int_{-\infty}^{\infty} \left(\frac{sq_e}{C}\right)^2 e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}} ds}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(sq_e)^2}{C\tau}} ds}$$

Note: *This approximation would not work if the energy due to one electron is comparable to τ in which case we would need to use the discrete summation.*

To simplify the problem we can change variables: $\sqrt{\frac{1}{2}\frac{(q_e s)^2}{C\tau}} = y \rightarrow s = y\frac{\sqrt{2C\tau}}{q_e}$

$$\langle V^2 \rangle = \frac{\int_{-\infty}^{\infty} \left(\frac{y\frac{\sqrt{2C\tau}}{q_e} q_e}{C}\right)^2 e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy} = \frac{\int_{-\infty}^{\infty} \left(\frac{y\sqrt{2C\tau}}{C}\right)^2 e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy} = \frac{2\tau}{C}\frac{\int_{-\infty}^{\infty} y^2 e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy}$$

The integral in the denominator is the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

The integral in the numerator can be calculated by parts:

$$\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \int_{-\infty}^{\infty} -(y/2) de^{-y^2} = -(y/2)e^{-y^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-y^2} d(y/2) = \frac{\sqrt{\pi}}{2}$$

With these results, the average of the voltage squares:

$$\left\langle V^2 \right\rangle = \frac{\tau}{C}, \quad \text{so} \quad \sqrt{\left\langle V^2 \right\rangle} = \sqrt{\frac{\tau}{C}}$$

With the values of the problem:

$$\sqrt{\left\langle V^2 \right\rangle} = \sqrt{\frac{(1.38\times10^{-23}\text{J/K})(300\text{K})}{47\times10^{-12}\text{F}}} = \mathbf{9.3\ \mu V}$$