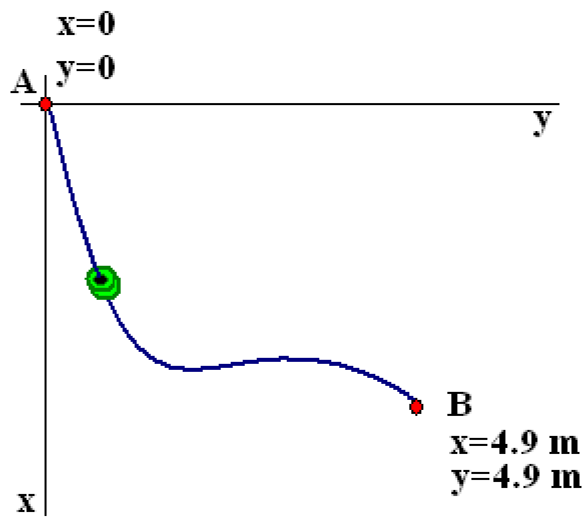
Classical Mechanics

Brachistochrone

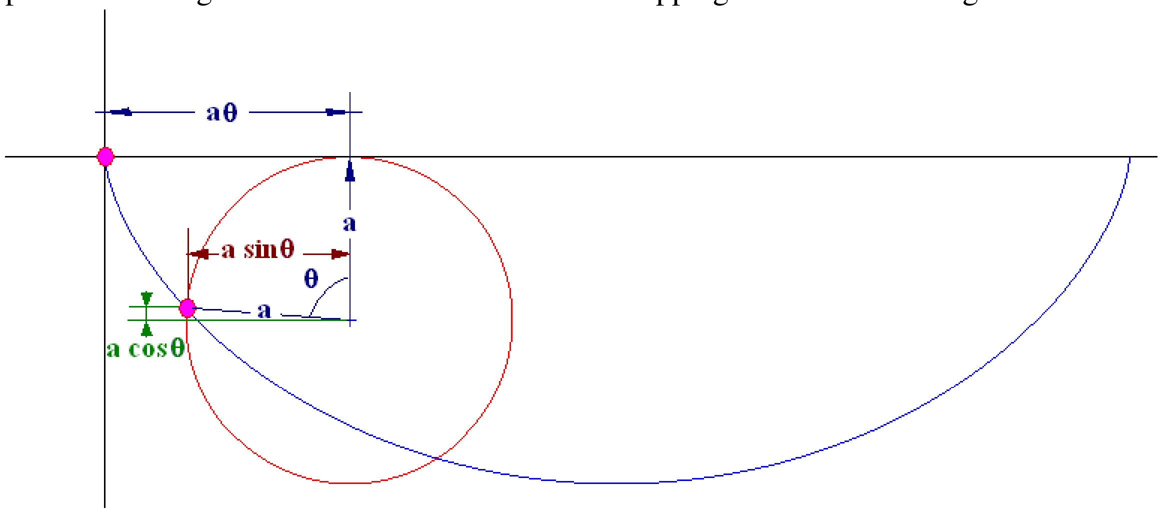
Problem 1.- Calculate the minimum time that will take a bead to get from point A to point B by starting from rest at point A and sliding without friction towards B accelerated by gravity ($g=9.8m/s^2$). Compare that time to the time it would take to get from A to B in a straight path.



Solution: It can be demonstrated that the shortest time between points A and B is by following a brachistochrone whose parametric equations are:

$$x = a(1-\cos\theta)$$
$$y = a(\theta - \sin\theta)$$

You can make sense of these equations by realizing that they describe the motion followed by a point on the edge of a circle that rotates without slipping as shown in the figure:



The problem gives us initial conditions: $x_1 = 0, y_1 = 0$ and final conditions: $x_2 = 4.9m, y_2 = 4.9m$. This implies that the initial angle is $\theta_1 = 0$ and the final angle can be calculated as follows:

$$4.9m = a(1-\cos\theta_2) \quad \text{and} \quad 4.9m = a(\theta_2 - \sin\theta_2)$$

dividing one equation by the other we eliminate "a" and we get:

$$1 = \frac{1-\cos\theta_2}{\theta_2 - \sin\theta_2} \rightarrow \theta_2 = 1 + \sin\theta_2 - \cos\theta_2$$

We can solve this equation using the technique of successive approximations and get:

$$\theta_2 = 2.4120$$

To find "a" we put the result above in the first equation and get:

$$4.9m = a(1-\cos(2.4120)) \rightarrow a = 2.8073m$$

To find the time we need to integrate the equation:

$$time = \int_{x_1}^{x_2} \left(\frac{1+y'^2}{2gx} \right)^{1/2} dx$$

But notice that $dx = \frac{dx}{d\theta} d\theta$ and $y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, so the integral can be converted into an integral over θ as follows:

$$dx = (a\sin\theta)d\theta \qquad\qquad y' = \frac{a(1-\cos\theta)}{(a\sin\theta)} = \frac{1-\cos\theta}{\sin\theta} = \tan(\theta/2)$$

With these substitutions:

$$time = \int_{\theta_1}^{\theta_2} \left(\frac{1+\tan^2(\theta/2)}{2ga(1-\cos\theta)} \right)^{1/2} a\sin\theta d\theta = \sqrt{\frac{a}{2g}} \int_{\theta_1}^{\theta_2} \left(\frac{1+\tan^2(\theta/2)}{1-\cos\theta} \right)^{1/2} \sin\theta d\theta$$

With the trigonometric identities $1+\tan^2(\theta/2) = \sec^2(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$:

$$time = \sqrt{\frac{a}{2g}} \int_{\theta_1}^{\theta_2} \left(\frac{\sec^2(\theta/2)}{2\sin^2(\theta/2)} \right)^{1/2} \sin\theta d\theta = \sqrt{\frac{a}{g}} \int_{\theta_1}^{\theta_2} \frac{\sec(\theta/2)}{2\sin(\theta/2)} \sin\theta d\theta = \sqrt{\frac{a}{g}} \int_{\theta_1}^{\theta_2} d\theta = \sqrt{\frac{a}{g}} \theta_2$$

This is a great simplification, and the time, given the values of the problem, is:

$$time = \sqrt{\frac{2.8073m}{9.8m/s^2}}\,2.4120 = \mathbf{1.29\ s}$$

As expected, this is shorter than the time it takes to get from A to B in a straight path, which you can calculate to be 1.41s.