Classical Mechanics

Newton's Laws

Problem 1.- An experiment reveals that the velocity of a particle follows the equation:

$v = \frac{A}{x^3}$, where A is a positive constant.

Calculate the net force acting on the particle as a function of x if its mass is "m".

Solution:

Since the force is equal to ma, we just need to find the acceleration:

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{A}{x^3}\frac{d\left(\frac{A}{x^3}\right)}{dx} = \frac{A}{x^3}\left(-\frac{3A}{x^4}\right) = -\frac{3A^2}{x^7}$$

So the force is: $$F = -\frac{3mA^2}{x^7}$$

Problem 1a.- An experiment reveals that the velocity of a particle of mass "m" follows the equation:

$v = \frac{A}{x^4}$, where A is a positive constant.

Calculate the net force acting on the particle as a function of x.

Solution: Since the force is equal to ma, we just need to find the acceleration:

$$a = \frac{dv}{dt} = \frac{d\left(\frac{A}{x^4}\right)}{dt}$$

But notice that we don't have v as a function of time, so we need to use the "chain rule" of calculus:

$$a = \frac{d\left(\frac{A}{x^4}\right)}{dx} = -\frac{4A}{x^5}\frac{dx}{dt} = -\frac{4A}{x^5}v = -\frac{4A^2}{x^9}$$

So the force is:

$$F = -\frac{4mA^2}{x^9}$$

Problem 2.- A skydiver jumps from a helicopter with zero initial velocity. In addition to the gravitational force –mg, she experiences an air resistance force equal to –bv, where b is a constant.

a) Find the terminal velocity of the skydiver.
b) Calculate the kinetic energy as a function of time.
c) Calculate the power dissipated on the skydiver as a function of time.

Solution:

a) The skydiver will reach terminal velocity when the net force acting on him is zero:

$$-\mathrm{mg}-\mathrm{bv}=0 \rightarrow \mathrm{v}=-\frac{\mathrm{mg}}{\mathrm{b}}$$

b) Let's calculate the velocity as a function of time:

$$\mathrm{F}=-\mathrm{mg}-\mathrm{bv} \rightarrow \mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}}=-\mathrm{mg}-\mathrm{bv} \rightarrow \frac{\mathrm{dv}}{\mathrm{v}+\mathrm{mg/b}}=-(\mathrm{b/m})\mathrm{dt}$$

Integrating we get:

$$\ln(\mathrm{v}+\mathrm{mg/b})=-(\mathrm{b/m})\mathrm{t}+\mathrm{C} \rightarrow \mathrm{v}=-\mathrm{mg/b}+\mathrm{C'e}^{-\mathrm{bt/m}}$$

Since the initial velocity is zero, C'=mg/b, so

$$\mathrm{v}=\mathrm{mg/b}(\mathrm{e}^{-\mathrm{bt/m}}-1)$$

The kinetic energy will then be:

$$\mathrm{K.E.}=\frac{1}{2}\mathrm{mv}^2=\frac{\mathrm{m}^3\mathrm{g}^2}{2b^2}(\mathrm{e}^{-\mathrm{bt/m}}-1)^2$$

c) The power will be:

$$\mathrm{Power}=Fv=\frac{\mathrm{m}^2\mathrm{g}^2}{b}(1\text{-}\mathrm{e}^{-\mathrm{bt/m}})\mathrm{e}^{-\mathrm{bt/m}}$$

Notice that the maximum power happens at $t=\frac{m}{b}\ln(2)$ when the power reaches

$$\mathrm{Power}_{\max}=\frac{\mathrm{m}^2\mathrm{g}^2}{4b}$$

At that point the net force is half the weight and the velocity is half the terminal velocity.

Problem 3.- Careful measurements show that an object moving in very thick molasses has an acceleration given by:

$a = -bv^3$

Calculate the velocity as a function of time if the initial velocity is $v(0) = v_o$

Solution: By definition: $a = \frac{dv}{dt}$, so: $\frac{dv}{dt} = -bv^3$, separating variables: $-\frac{dv}{v^3} = bdt$

Integrating: $\frac{1}{2v^2} = bt + C$

Using the initial condition: $\frac{1}{2v_o^{\,2}} = C \rightarrow \frac{1}{2v^2} = bt + \frac{1}{2v_o^{\,2}}$,

Solving for v gives us: $v = \frac{v_o}{\sqrt{1+2v_o^{\,2}bt}}$

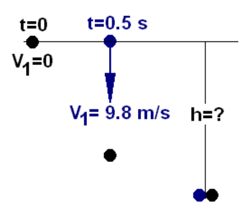
Problem 4.- A roller coaster uses magnetic brakes to slow down from an initial velocity of $v_1 = 14.7\,m/s$ to $v_2 = 0.735\,m/s$.

Calculate how long (time) it takes to do this, knowing that the acceleration can be written as: $a = -bv$, where $b = 1.5$ Hz

Solution:

$$\frac{dv}{dt} = -bv \rightarrow \frac{dv}{v} = -bdt \rightarrow \ln\left(\frac{v_2}{v_1}\right) = -bt \rightarrow t = \frac{1}{b}\ln\left(\frac{v_1}{v_2}\right) = \frac{1}{1.5}\ln\left(\frac{14.7}{0.735}\right) = \mathbf{2\ seconds}$$

Problem 5.- An object is dropped with an initial velocity zero. Half a second later another object is thrown downwards from the same initial point with a speed of 9.8m/s. Calculate how long (time) it will take the second object to reach the first.



Solution: The distance covered by the first object is $x = v_1 t + \frac{1}{2}at^2 = 4.9t^2$,

the second object starts 0.5 seconds later, so the time is only *t*-0.5 and the value of *x* is:

$$x = v_1 t + \frac{1}{2}at^2 = 9.8(t - 0.5) + 4.9(t - 0.5)^2,$$

but both numbers should be equal, so:

$$4.9t^2 = 9.8(t - 0.5) + 4.9(t - 0.5)^2$$

Solving for t we get: $0 = 9.8t - 4.9 - 4.9t + 1.175 \rightarrow t = \dfrac{4.9 - 1.175}{4.9} = \mathbf{0.75\ s}$