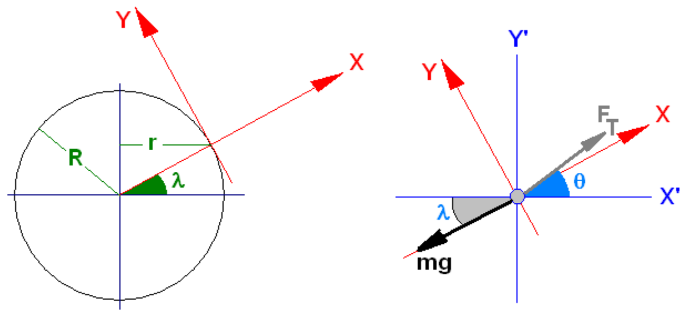
Classical Mechanics

Non-Inertial Frames of Reference

Problem 1.- Calculate the angle between a plumb line and the true vertical line in Lawton, OK, given that it is located at latitude 34.51° north.

Solution: There are two forces acting of the pendulum bob: The weight and the tension in the string as shown in the following schematic:



The two forces are in equilibrium in the Y' direction, so:

$$mg\sin\lambda = F_T\sin\theta$$

But in the X' direction the bob is accelerated towards the Earth's axis of rotation, so:

$$mg\cos\lambda - F_T\cos\theta = ma_R = m\omega^2 r$$

Replacing the first equation in the second (to eliminate F_T) we get:

$$mg\cos\lambda - \frac{mg\sin\lambda}{\sin\theta}\cos\theta = m\omega^2 r \rightarrow \cos\lambda - \sin\lambda\cot\theta = \frac{\omega^2 r}{g}$$

But $r = R\cos\lambda$, so: $\cos\lambda - \sin\lambda\cot\theta = \dfrac{\omega^2 R\cos\lambda}{g} \rightarrow \cot\theta = \left(1-\dfrac{\omega^2 R}{g}\right)\cot\lambda$

The angle that the plumb line makes with the true vertical is:

$$\theta - \lambda = \cot^{-1}\left[\left(1-\frac{\omega^2 R}{g}\right)\cot\lambda\right] - \lambda$$

With the values $\lambda = 34.51°$, $R = 6.38\times10^6 m$ and $\omega = \dfrac{2\pi}{day} = 7.27\times10^{-5} rad/s$ we get:

$$\theta - \lambda = \cot^{-1}\left[\left(1-\frac{(7.25\times10^{-5} rad/s)^2(6.38\times10^6 m)}{9.8 m/s^2}\right)\cot 34.51^\circ\right] - 34.51^\circ = \mathbf{0.092°}$$

Problem 1a.- Calculate the angle between a plumb line and the true vertical line given that the latitude is 30º south.

Solution: Using the results of the previous problem with the values $\lambda = \text{-}30^{\circ}$, $R = 6.38\times10^{6}m$ and $\omega = \dfrac{2\pi}{day} = 7.27\times10^{-5}\,rad/s$ we get:

$$\theta - \lambda = \cot^{-1}\left[\left(1 - \frac{(7.25\times10^{-5}\,rad/s)^2(6.38\times10^{6}m)}{9.8m/s^2}\right)\cot(-30^{\circ})\right] + 30^{\circ} = \mathbf{-0.085^{\circ}}$$

Problem 2.- A particle is fired upward in Lawton, OK, (located at latitude 34.51º north) and reaches an altitude h and falls back. Neglecting air resistance, calculate how far from the original position it will land.

Solution: In a rotating frame of reference you can still use Newton's second law of motion, but you need to add two additional forces due to rotation:

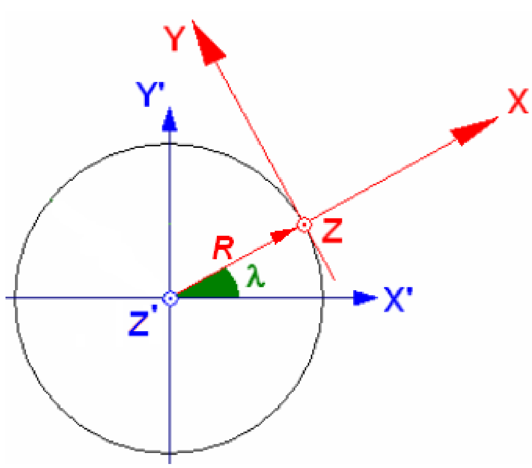
Centrifugal force: $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

Coriolis force: $-2m\vec{\omega}\times\vec{v}_r$

And another force due to the acceleration of the origin:

Acceleration of origin force: $-m\ddot{\vec{R}}$

In the present problem, the geometry looks as follows:



In this schematic the positive Y direction corresponds to north, the positive Z direction is west and the Y' direction corresponds to Earth's axis of rotation.
In a first approximation of the trajectory of the projectile we can ignore that the frame of reference is not inertial. Then the solution would be the usual parabola:

$\vec{r} = \left(u_o t - \frac{1}{2} g t^2, 0, 0 \right)$ where u_o is the initial velocity in the vertical direction. The only force considered here is the force due to the weight of the projectile.

Now we consider the additional forces, taking into account that:

$\vec{\omega} = (\omega \sin\lambda, \omega \cos\lambda, 0)$ and $\vec{v}_r = \frac{d\vec{r}}{dt} = (u_o - gt, 0, 0)$

Centrifugal force and acceleration of the origin: These two forces can be combined into a centrifugal force directed away from the axis of rotation of the Earth. If the initial trajectory is in the true vertical direction, this will cause the projectile to deviate towards the south. But normally when firing straight up this is already taken into account by pointing not to the true vertical, but to the "deviated" direction given by a plumb line (see the previous problem). The trajectory then is not affected much for small values of *h*.

Coriolis force: The force is given by:

$$F_{coriolis} = -2m\vec{\omega} \times \vec{v}_r = -2m(\omega \sin\lambda, \omega \cos\lambda, 0) \times (u_o - gt, 0, 0) = -2m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega \sin\lambda & \omega \cos\lambda & 0 \\ u_o - gt & 0 & 0 \end{vmatrix}$$

$$F_{coriolis} = (0, 0, 2m\omega \cos\lambda (u_o - gt))$$

We see that the Coriolis force in this case is a force in the positive Z direction (the west). We can integrate the acceleration in that direction to get the velocity and displacement:

$$a_{Coriolis} = \frac{F_{Coriolis}}{m} = 2\omega \cos\lambda (u_o - gt)$$

$$v_{Coriolis} = \int a_{Coriolis} dt = 2\omega \cos\lambda (u_o t - g t^2 / 2) + C_1$$

$$r_{Coriolis} = \int v_{Coriolis} dt = 2\omega \cos\lambda (u_o t^2 / 2 - g t^3 / 6) + C_1 t + C_2$$

The two integration constants are zero since the initial velocity in Z direction and the initial displacement are zero. So this last equation tells us that the projectile will land to the west of the original position, the distance being:

$$Deviation = 2\omega \cos\lambda (u_o t^2 / 2 - g t^3 / 6)$$

Where *t* is the total time of the trajectory, which time can be found in terms of *h*:

$$h = \frac{1}{2} g (t/2)^2 \rightarrow t = \sqrt{\frac{8h}{g}} \quad and \quad u_o = \sqrt{2gh} \rightarrow Deviation = \frac{4}{3} \omega \sqrt{\frac{8h^3}{g}} \cos\lambda$$

Problem 2a.- A bullet is aimed and fired directly north in Lawton, OK. It reaches a wall, which is at a distance d=350m away. Neglecting air resistance, calculate how far from the original target it will hit. Consider both the "drop" due to gravity and the Coriolis Effect.

Assume the initial velocity of the bullet is horizontal and v=560m/s.

Solution: In a rotating frame of reference, you can still use Newton's second law of motion, but you need to add two additional forces due to rotation:

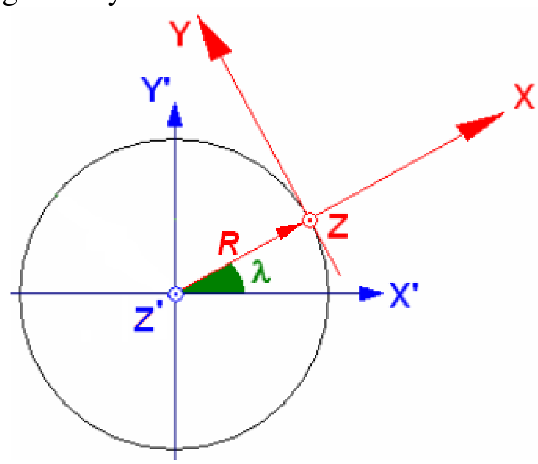
Centrifugal force: $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

Coriolis force: $-2m\vec{\omega}\times\vec{v}_r$

And another force due to the acceleration of the origin:

Acceleration of origin force: $-m\ddot{\vec{R}}$

In the present problem, the geometry looks as follows:



In this schematic the positive Y direction corresponds to north, the positive Z direction is west and the Y' direction corresponds to the axis of the Earth.

In a first approximation of the trajectory of the projectile we can ignore that the frame of reference is not inertial. Then the solution would be the usual parabola that you learned about in Physics I:

$\vec{r} = \left(-\frac{1}{2}gt^2, u_o t, 0\right)$ where u_o is the initial velocity (to the North). The only force considered here is the force due to the weight of the projectile, which produces a drop of:

$$drop = -\frac{1}{2}gt^2 = -\frac{1}{2}\times 9.8\times\left(\frac{350}{560}\right)^2 = \mathbf{-1.91m}$$

Now we consider the additional forces, taking into account that:

$\vec{\omega} = (\omega\sin\lambda, \omega\cos\lambda, 0)$ and $\vec{v}_r = \frac{d\vec{r}}{dt} = (-gt, u_o, 0)$

Centrifugal force and acceleration of the origin: These two forces can be combined into a centrifugal force directed away from the axis of rotation of the Earth. If the initial trajectory is in the true horizontal direction to the north this will cause the projectile to deviate. But normally when firing horizontally this is already taken into account by pointing not using the true vertical, but to the "deviated" direction given by a plumb line as the reference. The trajectory then is not affected much.

Coriolis force: The force is

$$F_{coriolis} = -2m\vec{\omega}\times\vec{v}_r = -2m(\omega\sin\lambda, \omega\cos\lambda, 0)\times(-gt, u_o, 0) = -2m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega\sin\lambda & \omega\cos\lambda & 0 \\ -gt & u_o & 0 \end{vmatrix}$$

$$F_{coriolis} = (0,0,-2m\omega(u_o\sin\lambda + gt\cos\lambda))$$

We see that the Coriolis force in this case is a force in the negative Z direction (the east). We can integrate the acceleration in that direction to get the velocity and displacement:

$$a_{Coriolis} = \frac{F_{coriolis}}{m} = -2\omega(u_o\sin\lambda + gt\cos\lambda)$$

$$v_{Coriolis} = \int a_{Coriolis}dt = -2\omega(u_o\sin\lambda t + gt^2\cos\lambda/2) + C_1$$

$$r_{Coriolis} = \int v_{Coriolis}dt = -2\omega(u_o\sin\lambda t^2/2 + gt^3\cos\lambda/6) + C_1t + C_2$$

The two integration constants are zero since the initial velocity in Z direction and the initial displacement are zero. So this last equation tells us that the projectile will land to the east of the original position, the distance being:

$$Deviation = 2\omega(u_o\sin\lambda t^2/2 + gt^3\cos\lambda/6)$$

Where t is the total time of the trajectory, which time can be found:

$$t = \frac{350m}{560m/s} = 0.625 \rightarrow \ Deviation = \mathbf{9.4\ mm}$$

Problem 3.- Calculate in what direction and by how much we will miss a target if we aim towards the north at a level point located 350 away. The speed of the bullet is 700 m/s.
Consider both, the drop due to gravity and the Coriolis Effect.
Take the latitude to be 34.5º

Solution: To find how much the bullet will drop we use:

$$\mathrm{y} = \frac{1}{2}\mathrm{gt}^2 = \frac{1}{2}\mathrm{g}\left(\frac{350}{700}\right)^2 = \mathbf{1.23\ m\ (drop)}$$

To find the effect of the Coriolis force notice that the deflection will be towards the east and the acceleration will be:

$$a_{Coriolis} = 2\omega v \sin\lambda = 2\times\frac{2\pi}{86,400}\times 700\times\sin 34.5^{\circ} = 0.057 m/s^2$$

And the deflection will be:

$$x = \frac{1}{2}a_{Coriolis}t^2 = \frac{1}{2}0.057 m/s^2\times\left(\frac{350m}{700m/s}\right)^2 = \mathbf{0.0072\ m\ (east)}$$

Problem 4.- The transformation of the coordinates of a point $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ into a rotated frame of reference is given as follows: $\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 0.8 & 0 & 0.6 \\ 0 & 1 & 0 \\ -0.6 & 0 & 0.8 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

Identify the axis of rotation and the angle.

Solution: The axis of rotation is the y-axis and the angle is 37 degrees.

Problem 5.- A car is at latitude 45 degrees south and going towards the north. In what direction would be the Coriolis force acting on the car?

Solution: The Coriolis force will be towards the west.