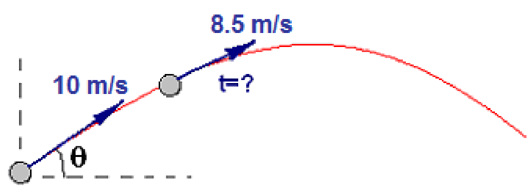
Classical Mechanics

Projectiles

Problem 1.- A projectile is launched at an angle $\theta = 36^\circ$ with an initial speed $v_1 = 10\text{m/s}$. Calculate how long later the speed will be only 8.5 m/s.



Solution: The initial velocities are:

$$V_{1x} = 10\cos 36^\circ = 8.09 \text{ m/s}$$
$$V_{1y} = 10\sin 36^\circ = 5.88 \text{ m/s}$$

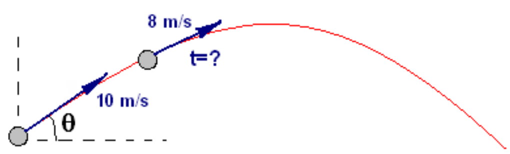
Later, when the speed is 8.5 m/s the horizontal velocity will stay at 8.09 m/s, but the vertical velocity will be:

$$V_{2y} = \sqrt{8.5^2 - 8.09^2} = 2.61 m/s$$

Now we can find the time:

$$t = \frac{v_{2y} - v_{1y}}{a_y} = \frac{2.61 - 5.88}{-9.8} = \mathbf{0.033\ s}$$

Problem 1a.- A projectile is launched at an angle $\theta = 53^\circ$ with an initial speed $v_1 = 10\text{m/s}$. Calculate how long later the speed will be only 8 m/s.



Solution: The initial velocities are:

$$V_{1x} = 10\cos 53^\circ = 6 \text{ m/s}$$
$$V_{1y} = 10\sin 53^\circ = 8 \text{ m/s}$$

Later, when the speed is 8m/s the horizontal velocity will stay at 6 m/s, but the vertical velocity will be:

$$V_{2y} = \sqrt{8^2 - 6^2} = 5.3 m/s$$

Now we can find the time:

$$t = \frac{v_{2y} - v_{1y}}{a_y} = \frac{5.3 - 8}{-9.8} = \mathbf{0.28\ s}$$

Problem 2.- A rocket is launched straight up with the following characteristics:

Initial mass: $m_i = 2.8 \times 10^6 kg$

Final mass: $m_f = 0.7 \times 10^6 kg$

Fuel mass: $m_{fuel} = m_i - m_f = 2.1 \times 10^6 kg$

Relative velocity of gasses: $v_r = 2{,}600 m/s$

Thrust: $F_{thrust} = 37 \times 10^6 N$

Determine the rate of fuel burning, the total time until the fuel is consumed, the acceleration, velocity, and height as a function of time. For these calculations assume that the value of g stays constant ($9.8 m/s^2$).

Solution:

Rate of fuel burning: $\frac{dm}{dt} = -\frac{F_{thrust}}{v_r} = -\frac{37 \times 10^6 N}{2{,}600 m/s} = 14{,}231 kg/s$

Total time: $t = \frac{m_{fuel}}{\left|\frac{dm}{dt}\right|} = \frac{2.1 \times 10^6 kg}{14{,}231 kg/s} = 147.6 s$

Acceleration: $a = \frac{\sum F}{m} = \frac{37 \times 10^6}{2.8 \times 10^6 - 14{,}231 t} - 9.8$

$$v = \int_0^t \left(\frac{37 \times 10^6}{2.8 \times 10^6 - 14{,}231 t} - 9.8 \right) dt$$

$$v = \int_0^t \left(\frac{2600}{196.7 - t} - 9.8 \right) dt = 2600 \ln\left(\frac{196.7}{196.7 - t} \right) - 9.8t$$

$$x = \int_0^t \left(2600 \ln\left(\frac{196.7}{196.7 - t} \right) - 9.8t \right) dt$$

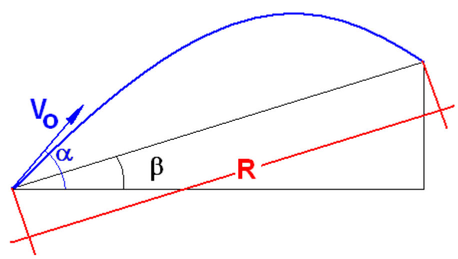
$$x = -2600 \int_0^t \ln\left(1 - \frac{t}{196.7} \right) dt - 4.9t^2$$

$$x = 2600 \times 196.7 \int_1^{1-\frac{t}{196.7}} \ln u du - 4.9t^2$$

$$x = 2600 \times 196.7 (u \ln u - u)\Big|_1^{1-\frac{t}{196.7}} - 4.9t^2$$

$$x = 2600\left[(196.7 - t)\ln\left(1 - \frac{t}{196.7}\right) + t\right] - 4.9t^2$$

Problem 3.- A projectile is shot up a slope (inclined an angle β) with an initial speed V_o. Determine the angle α that will produce the maximum range R.



Solution: The problem is to find the value of α that gives the maximum range. To do this notice that the equations of motion are:

$x = V_o(\cos\alpha)t$ and $y = V_o(\sin\alpha)t - \frac{1}{2}gt^2$

We know that the projectile will hit the ground when $\tan\beta = \frac{y}{x}$, so we can calculate the time of flight of the particle:

$$\tan\beta = \frac{V_o(\sin\alpha)t - \frac{1}{2}gt^2}{V_o(\cos\alpha)t} \rightarrow t = 2\frac{V_o\cos\alpha}{g}(\tan\alpha - \tan\beta)$$

The range can be calculated with the equation: $R = \frac{x}{\cos\beta}$, so:

$$R = \frac{V_o\cos\alpha}{\cos\beta}t = \frac{V_o\cos\alpha}{\cos\beta}\frac{2V_o\cos\alpha(\tan\alpha - \tan\beta)}{g} = \frac{2V_o^2\cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta)$$

If we want the best range, the derivative of R with respect to α should be zero, so:

$$\frac{\partial}{\partial\alpha}\left(\frac{2V_o^2\cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta)\right) = 0 \rightarrow \frac{\partial}{\partial\alpha}\left((\tan\alpha - \tan\beta)\cos^2\alpha\right) = 0$$

Taking the derivative:

$$\frac{\partial}{\partial\alpha}\left((\tan\alpha-\tan\beta)\cos^2\alpha\right)=\frac{1}{\cos^2\alpha}\cos^2\alpha-(\tan\alpha-\tan\beta)2\cos\alpha\sin\alpha=0$$
$$\rightarrow(\tan\alpha-\tan\beta)\sin 2\alpha=1$$

This last equation can be simplified as follows:

$$(\tan\alpha-\tan\beta)=\frac{1}{\sin 2\alpha}$$
$$\tan\beta=\tan\alpha-\frac{1}{\sin 2\alpha}=\frac{2\sin^2\alpha-1}{2\sin\alpha\cos\alpha}=-\frac{\cos 2\alpha}{\sin 2\alpha}=-\cot 2\alpha$$

Using the equation: $\tan(-\beta)=-\tan\beta$ we get:

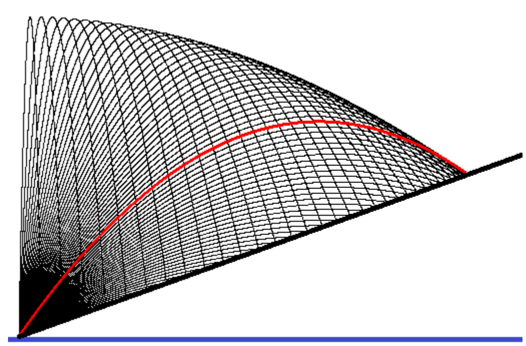
$$\tan(-\beta)=\cot 2\alpha$$

And since $\cot 2\alpha=\tan(90-2\alpha)$ we get:

$$\tan(-\beta)=\tan(90-2\alpha)\rightarrow-\beta=90-2\alpha\rightarrow$$

$$\alpha=45^o+\beta/2$$

For example, for β=20°, the best angle is α = 45° + 20°/2 = 55°, which is highlighted in red in the following figure:



Interestingly, the solution also works for negative slopes, for example for $\beta = -20°$, the best angle is $\alpha = 45° + (-20°)/2 = 35°$, which is highlighted in red in the following figure: