Classical Mechanics

Damped oscillator

Problem 1.- Prove that for a lightly damped oscillator, the change in frequency caused by the damping is approximately $\frac{\omega_o}{8Q^2}$. Based on that, if damping causes a 1% decrease in the frequency of an oscillator, what is its Q value?

Solution:

a) ***Proof:***
When the system is lightly damped, the angular frequency drops to the value given by:

$$\omega_1 = \sqrt{\omega_o^{\ 2} - \beta^2}$$

We can rearrange the equation as follows: $\omega_1 = \omega_o \sqrt{1 - \frac{\beta^2}{\omega_o^{\ 2}}}$

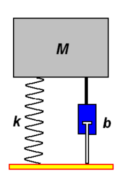
If the system is truly lightly damped, the ratio $\frac{\beta^2}{\omega_o^{\ 2}}$ is small, so we can approximate the radical by: $\omega_1 = \omega_o \left(1 - \frac{\beta^2}{2\omega_o^{\ 2}}\right)$ so the change in frequency is $\omega_o - \omega_1 = \omega_o \left(\frac{\beta^2}{2\omega_o^{\ 2}}\right)$

But by definition $Q = \frac{\omega_R}{2\beta} \approx \frac{\omega_o}{2\beta} \rightarrow \beta \approx \frac{\omega_o}{2Q}$

With this change of variable, we get: $\omega_o - \omega_1 = \omega_o \left(\frac{\left(\frac{\omega_o}{2Q}\right)^2}{2\omega_o^{\ 2}}\right) = \frac{\omega_o}{8Q^2}$

b) ***1% decrease in the frequency:*** It means that $\frac{1}{8Q^2} = 0.01$, so $Q = \sqrt{\frac{1}{0.08}} =$ **3.5**

Problem 2.- Calculate the position x(t) of the critically damped oscillator shown in the figure whose mass is M=10kg and its spring constant k=40N/m. Consider that x is measured with respect to the equilibrium condition and the initial velocity v(0)=0 and initial position x(0)=1.5 m



Solution: The differential equation is: $\ddot{x}+2\beta\dot{x}+\omega_o^{\ 2}=0$

If the oscillator is critically damped, it means that: $\beta=\omega_o=\sqrt{\frac{k}{m}}=\sqrt{\frac{20}{5}}=2 rad/s$

The solution has the form: $x=Ae^{-2t}+Bte^{-2t}$

The initial conditions will give us the constants ***A*** and ***B***:

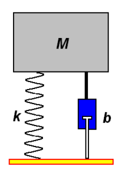
$x(0)=1.5=A$

and

$\dot{x}(0)=-2Ae^{-2t}+Be^{-2t}-2Bte^{-2t}=-2A+B-2Bt=-2A+B=0\rightarrow B=3$

$x=1.5e^{-2t}+3te^{-2t}$

Problem 2a.- Calculate the position x(t) of the critically damped oscillator shown in the figure whose mass is M=5kg and its spring constant k=20N/m. Consider that x is measured with respect to the equilibrium condition and the initial velocity v(0)=0 and initial position x(0)=1.5 m



Solution:

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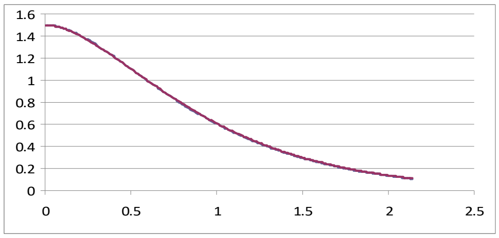
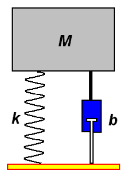


Problem 2c.- Calculate the free response of the oscillator shown in the figure. Indicate x(t) as a function of time as your answer. Consider that x is measured with respect to the equilibrium condition and the initial velocity v=6m/s and initial position x=0

M=5kg, k=5 N/m and b=1 Ns/m



Solution: $\ddot{x}+2\beta\dot{x}+\omega_\circ^2=0$

$$\beta=\frac{b}{2M}=\frac{1}{2\times 5}=0.1$$

$\omega_1=\sqrt{\omega_\circ^2-\beta^2}=\sqrt{1-0.1^2}=0.995$ …. Under-damped case.

$$x=Ae^{-0.1t}\sin(0.995t)+Be^{-0.1t}\cos(0.995t)$$

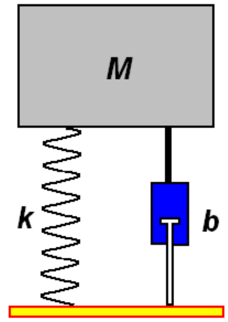
Initial conditions:

$$x(0)=B=0\rightarrow B=0$$
$$\dot{x}(t)=-0.1Ae^{-0.1t}\sin(0.995t)+Ae^{-0.1t}\cos(0.995t)$$
$$\dot{x}(0)=A0.995=6\rightarrow A=6.03$$

Solution: $x=6.03e^{-0.1t}\sin(0.995t)$

Problem 2d.- Calculate the free response of the oscillator shown in the figure. Indicate x(t) as a function of time as your answer.

Consider that x is measured with respect to the equilibrium condition and the initial velocity v=0.5m/s and x(0)=1m



a) for M=2.5kg, k=10 N/m and b=5 Ns/m

b) for M=2.5kg, k=10 N/m and b=10 Ns/m

c) for M=2.5kg, k=10 N/m and b=20 Ns/m

Solution:

a) $\ddot{x}+2\beta\dot{x}+\omega_{\circ}^{2}=0$

$$\beta=\frac{b}{2M}=\frac{5}{2\times 2.5}=1rad/s$$

$\omega_1=\sqrt{\omega_{\circ}^{2}-\beta^2}=\sqrt{4-1^2}=1.73rad/s$ …. Under-damped case.

$$x=Ae^{-t}\sin(1.73t)+Be^{-t}\cos(1.73t)$$

Initial conditions:

$$x(0)=B=1\rightarrow B=1$$

$$\dot{x}(t)=-Ae^{-t}\sin(1.73t)-Be^{-t}\cos(1.73t)+1.73Ae^{-t}\cos(1.73t)-1.73Be^{-t}\sin(1.73t)$$

$$\dot{x}(0)=-B+1.73A=0.5\rightarrow A=\frac{0.5+1}{1.73}=0.866$$

Solution: $x=0.866e^{-t}\sin(1.73t)+e^{-t}\cos(1.73t)$

b) $\ddot{x}+2\beta\dot{x}+\omega_{\circ}^{2}=0$

$$\beta=\frac{b}{2M}=2rad/s$$

$\omega_1=\sqrt{\omega_{\circ}^{2}-\beta^2}=0$ …. Critically damped case

$$x=Ae^{-2t}+Bte^{-2t}$$

Initial conditions:

$$x(0) = A = 1$$
$$\dot{x}(t) = -2Ae^{-2t} - 2Bte^{-2t} + Be^{-2t}$$
$$\dot{x}(0) = -2A + B = 0.5 \rightarrow B = 2.5$$

Solution: $x = e^{-2t} + 2.5te^{-2t}$

c) $\ddot{x} + 2\beta\dot{x} + \omega_{\circ}^{2} = 0$

$$\beta = \frac{b}{2M} = \frac{20}{2\times 2.5} = 4rad/s$$

Over-damped case:

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_{\circ}^2} = -4 + \sqrt{12} = -0.535$$
$$r_2 = -\beta - \sqrt{\beta^2 - \omega_{\circ}^2} = -4 - \sqrt{12} = -7.464$$

$$x = Ae^{-0.535t} + Be^{-7.464t}$$

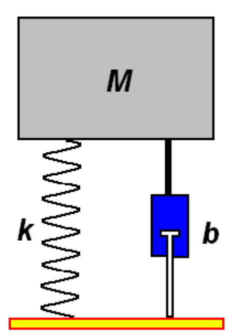
Initial conditions:

$$x(0) = A + B = 1$$
$$\dot{x}(0) = -0.535A - 7.464B = 0.5$$

So: $A = \dfrac{\begin{vmatrix} 1 & 1 \\ 0.5 & -7.464 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -0.535 & -7.464 \end{vmatrix}} = \dfrac{-7.464 - 0.5}{-7.464 + 0.535} = 1.15 \rightarrow B = -0.15$

Solution: $x = 1.15e^{-0.535t} - 0.15e^{-7.464t}$

Problem 2e.- Calculate the free response of the oscillator shown in the figure. Indicate x(t) as a function of time as your answer.
Consider that x is measured with respect to the equilibrium condition and the initial velocity v=0.15m/s and x(0)=0



a) for M=2.5kg, k=10 N/m and b=5 Ns/m
b) for M=2.5kg, k=10 N/m and b=10 Ns/m
c) for M=2.5kg, k=10 N/m and b=20 Ns/m

Solution:

a) $\ddot{x} + 2\beta\dot{x} + \omega_\circ^2 = 0$

$$\beta = \frac{b}{2M} = \frac{5}{2\times 2.5} = 1 rad/s$$

$\omega_1 = \sqrt{\omega_\circ^2 - \beta^2} = \sqrt{4 - 1^2} = 1.73 rad/s$ …. Under-damped case.

$$x = Ae^{-t}\sin(1.73t + \phi)$$

Initial conditions:

$$x(0) = A\sin(\phi) = 0 \rightarrow \phi = 0$$

$$\dot{x}(0) = 1.73A\cos(0) = 0.15 \rightarrow A = \frac{0.15}{1.73}$$

Solution: $x = 0.0866e^{-t}\sin(1.73t)$

b) $\ddot{x} + 2\beta\dot{x} + \omega_\circ^2 = 0$

$$\beta = \frac{b}{2M} = 2 rad/s$$

$\omega_1 = \sqrt{\omega_\circ^2 - \beta^2} = 0$ …. Critically damped case

$$x = Ae^{-2t} + Bte^{-2t}$$

Initial conditions:

$$x(0) = A = 0$$

$$\dot{x}(0) = B = 0.15$$

Solution: $x = 0.15te^{-2t}$

c) $\ddot{x} + 2\beta\dot{x} + \omega_\circ^2 = 0$

$$\beta = \frac{b}{2M} = \frac{20}{2\times 2.5} = 4 rad/s$$

Over-damped case:

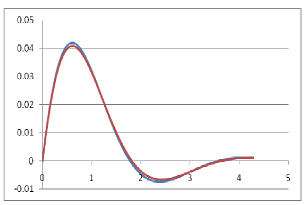
$$r_1 = -\beta + \sqrt{\beta^2 - \omega_o^2} = -4 + \sqrt{12} = -0.535$$
$$r_2 = -\beta - \sqrt{\beta^2 - \omega_o^2} = -4 - \sqrt{12} = -7.464$$
$$x = Ae^{-0.535t} + Be^{-7.464t}$$

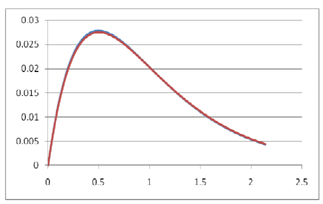
Initial conditions:

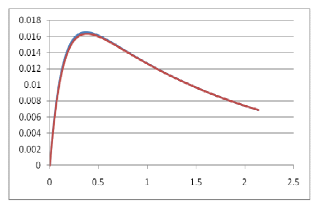
$$x(0) = A + B = 0$$
$$\dot{x}(0) = -0.535A - 7.464B = 0.15$$

So: $A = \dfrac{\begin{vmatrix} 0 & 1 \\ 0.15 & -7.464 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -0.535 & -7.464 \end{vmatrix}} = \dfrac{-0.15}{-7.464 + 0.535} = 0.0216$ and $B = -0.0216$

Solution: $x = 0.0216(e^{-0.535t} - e^{-7.464t})$







Problem 3.- A small fiber vibrates like a simple harmonic oscillator with light damping. In one experiment you determine that the maximum amplitude occurs when the driving frequency is f=134 kHz, but the angular delay between driving force and oscillation reaches 90º at f=136 kHz. Based on these results, what is the natural frequency of oscillation in the absence of damping and how much is β?

Solution: The angular delay between driving force and oscillation is given by:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

This value reaches 90º when $\omega_o = \omega$, so $\omega_o = 2\pi(136kHz) =$ **855 krad/s**

The resonance condition is $\omega = \omega_R = \sqrt{\omega_o^2 - 2\beta^2}$, so:

$$\beta = \sqrt{\frac{\omega_o^2 - \omega^2}{2}} = 2\pi\sqrt{\frac{136kHz^2 - 134kHz^2}{2}} = \textbf{104 krad/s}$$