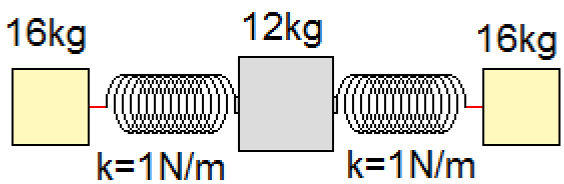
Classical Mechanics

Eigen frequencies

Problem 1.- Consider a model of a linear molecule as three masses connected with springs. Calculate the eigen frequencies of the system.



Solution: $L=\frac{1}{2}16\dot{x}_1{}^2+\frac{1}{2}12\dot{x}_2{}^2+\frac{1}{2}16\dot{x}_3{}^2-\frac{1}{2}1(x_2-x_1)^2-\frac{1}{2}1(x_3-x_2)^2$

$$L=8\dot{x}_1{}^2+6\dot{x}_2{}^2+8\dot{x}_3{}^2-\frac{1}{2}(x_2-x_1)^2-\frac{1}{2}(x_3-x_2)^2$$

Equations of motion:

$\frac{\partial L}{\partial \dot{x}_1}=16\dot{x}_1 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}=16\ddot{x}_1$	$\frac{\partial L}{\partial \dot{x}_2}=12\dot{x}_2 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}=12\ddot{x}_2$	$\frac{\partial L}{\partial \dot{x}_3}=16\dot{x}_3 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_3}=16\ddot{x}_3$
$\frac{\partial L}{\partial x_1}=x_2-x_1$	$\frac{\partial L}{\partial x_2}=x_1-x_2+x_3-x_2$	$\frac{\partial L}{\partial x_3}=x_2-x_3$
$16\ddot{x}_1=-x_1+x_2$	$12\ddot{x}_2=x_1-2x_2+x_3$	$16\ddot{x}_3=x_2-x_3$

With solutions of this form:

$$x_1=A_1e^{i\omega t},x_2=A_2e^{i\omega t},x_3=A_3e^{i\omega t}$$

We write the equations of motion as a matrix

$$\begin{bmatrix} 1/16 & -1/16 & 0 \\ -1/12 & 2/12 & -1/12 \\ 0 & -1/16 & 1/16 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}=\omega^2\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Finding the eigen frequencies:

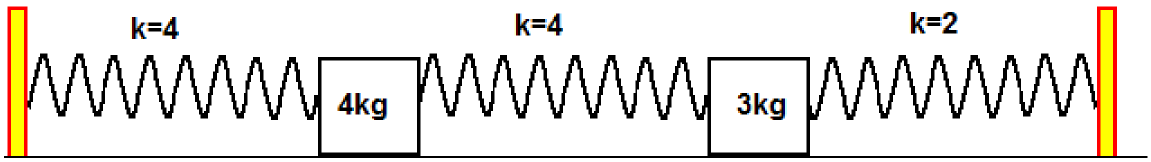
$$\det\begin{bmatrix} 1/16-\omega^2 & -1/16 & 0 \\ -1/12 & 2/12-\omega^2 & -1/12 \\ 0 & -1/16 & 1/16-\omega^2 \end{bmatrix}=0$$

$$\rightarrow (1/16-\omega^2)((2/12-\omega^2)(1/16-\omega^2)-(-1/16)(-1/12))+1/16(-1/12)(1/16-\omega^2)=0$$

$$\rightarrow (1-16\omega^2)\left[-44+192\omega^2\right]\omega^2=0$$

The three eigen frequencies will be $\omega=0,\ \omega=0.25\ and\ \omega=0.478$

Problem 2.- Find the eigen frequencies



There is no friction and the springs are un-stretched in the initial position.

Solution: The Lagrangian for the system is

$$L=\frac{1}{2}4\dot{x}_1{}^2+\frac{1}{2}3\dot{x}_2{}^2-\frac{1}{2}4x_1{}^2-\frac{1}{2}2x_2{}^2-\frac{1}{2}4(x_1-x_2)^2$$

After simplifying $L=2\dot{x}_1{}^2+1.5\dot{x}_2{}^2-4x_1{}^2-3x_2{}^2+4x_1x_2$

First, we find the equation of motion for x_1

$$\frac{\partial L}{\partial \dot{x}_1}=4\dot{x}_1 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}=4\ddot{x}_1$$

$$\frac{\partial L}{\partial x_1}=-8x_1+4x_2$$

$$\ddot{x}_1=-2x_1+x_2\ldots(1)$$

Similarly, we find the equation of motion for x_2

$$\frac{\partial L}{\partial \dot{x}_2}=3\dot{x}_2 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}=3\ddot{x}_2$$

$$\frac{\partial L}{\partial x_2}=4x_1-6x_2$$

$$3\ddot{x}_2=4x_1-6x_2\ldots(2)$$

Now that we have the two equations we try the solution $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$, so the equations become

$$-A\omega^2 = -2A + B$$
$$-B\omega^2 = 4/3A - 2B$$

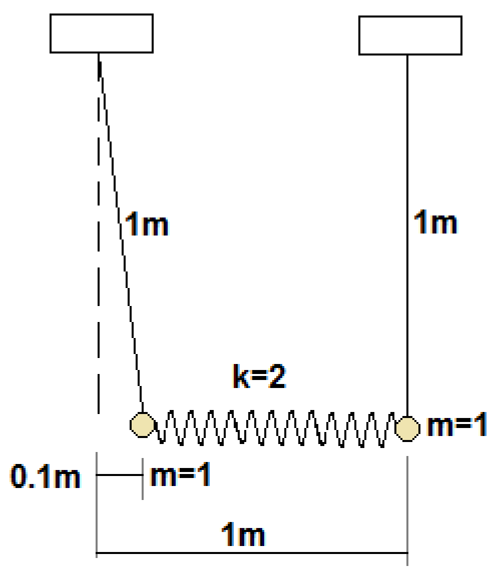
These equations can be written in matrix form as $\begin{bmatrix} -2 & 1 \\ 4/3 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = -\omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$

We find the eingen frequencies of the system as follows:

$$\det\begin{bmatrix} -2+\omega^2 & 1 \\ 4/3 & -2+\omega^2 \end{bmatrix} = 0 \rightarrow (\omega^2 - 2)^2 - 4/3 = 0 \rightarrow \omega = \sqrt{2 \pm \frac{2\sqrt{3}}{3}}$$

Therefore, the two angular frequencies are 1.776 and 0.919

Problem 3.- Solve for x_1 and x_2. The un-stretched length of the spring to be X_o=1m



Solution:

$$x_1 = A\cos(\sqrt{13.8}t) + A'\sin(\sqrt{13.8}t) + C\cos(\sqrt{9.8}t) + C'\sin(\sqrt{9.8}t)$$
$$x_2 = -A\cos(\sqrt{13.8}t) - A'\sin(\sqrt{13.8}t) + C\cos(\sqrt{9.8}t) + C'\sin(\sqrt{9.8}t)$$

Initial conditions:

$$x_1(0) = 0.1 = A + C$$
$$x_2(0) = 0 = -A + C$$
$$\dot{x}_1(0) = 0 = A'\sqrt{13.8} + C'\sqrt{9.8}$$
$$\dot{x}_2(0) = 0 = -A'\sqrt{13.8} + C'\sqrt{9.8}$$

Therefore, the solution will be

$$x_1 = 0.05\cos(\sqrt{13.8}t) + 0.05\cos(\sqrt{9.8}t)$$
$$x_2 = -0.05\cos(\sqrt{13.8}t) + 0.05\cos(\sqrt{9.8}t)$$