Classical Mechanics

General oscillations

Problem 1.- Consider a potential given by:

$V = \frac{1}{2}x^n$, where n is given

Calculate the period of oscillations for a 1-kg particle trapped in that potential that has total energy = 0.5 J.

Solution: The total energy is the sum of potential and kinetic energy:

$$T.E. = V + K.E = \frac{1}{2}x^n + \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2}{m}\left(T.E. - \frac{1}{2}x^n\right)}$$

But $v = \frac{dx}{dt} \rightarrow dt = \frac{dx}{v} = \frac{dx}{\sqrt{\frac{2}{m}\left(T.E. - \frac{1}{2}x^n\right)}} = \frac{dx}{\sqrt{\frac{2}{1}\left(0.5 - \frac{1}{2}x^n\right)}} = \frac{dx}{\sqrt{1-x^n}}$

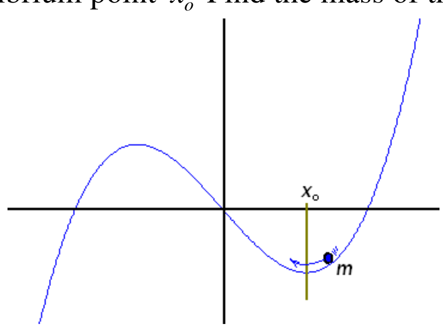
To find the period we can use the symmetry of the problem and calculate the time it takes to get from x=0 to x=1 (the turning point) and multiply by 4:

$$T = 4\int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

If n=2 the integral is simple, and for higher exponents we can use numerical techniques to get:

For n = 4, T= 5.24
For n = 6, T= 4.85
For n = 8, T= 4.65
For n = 10, T= 4.52
For n = 12, T= 4.44
For n = 14, T= 4.38

Problem 2.- A particle trapped in the potential: $\text{P.E.} = 3x^5 - 4x$ oscillates at $\omega = 0.55\,\text{rad/s}$ around the equilibrium point x_o Find the mass of the particle.



Solution: First we find the value of x_o:

$$\frac{dP.E.}{dx} = 15x^4 - 4 = 0 \rightarrow x_o = \sqrt[4]{\frac{4}{15}}$$

Then the value of the "k" constant is: $"k" = \frac{d^2P.E.}{dx^2} = 60x^3 = 60\left(\frac{4}{15}\right)^{3/4}$

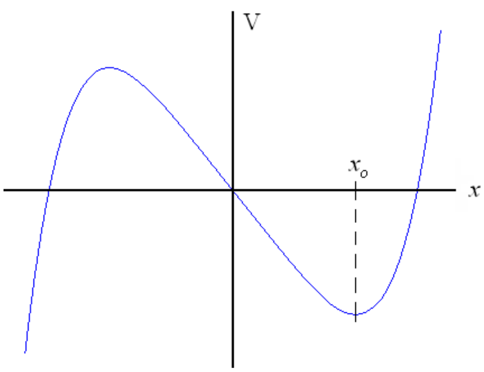
The value of $\omega = 0.55\text{rad/s}$ can be written as:

$$\omega = \sqrt{\frac{k}{m}} \rightarrow m = \frac{k}{\omega^2} = \frac{60\left(\frac{4}{15}\right)^{3/4}}{0.55^2} = \mathbf{73.6\ kg}$$

Problem 2a.- A particle of mass *m* is trapped in the potential:

$V = ax^5 - bx$

Where a and b are positive constants. Find the value of x_o, where the potential has a local minimum and find the angular frequency of small oscillations of the particle around that point.



Solution: We first find the minimum point:

$$\frac{dV}{dx} = 0 = 5ax^4 - b \rightarrow x_o = \sqrt[4]{\frac{b}{5a}}$$

Then we expand the potential $V = V(x_o) + \left.\frac{dV}{dx}\right|_{x_o}(x - x_o) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_o}(x - x_o)^2 + \ldots$

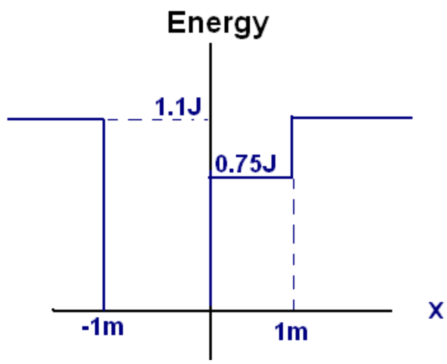
We notice that:

$$\left.\frac{d^2V}{dx^2}\right|_{x_o} = 20ax_o^3 = 20a\left(\sqrt[4]{\frac{b}{5a}}\right)^3$$

Since it is a positive number, it will dominate small oscillations and the motion will be approximately simple harmonic with angular frequency:

$$\omega_o = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_o}}{m}} = \sqrt{\frac{20a\left(\sqrt[4]{\frac{b}{5a}}\right)^3}{m}} = \sqrt{\frac{4}{m}(5ab)^{3/4}}$$

Problem 3.- A particle of mass 2kg and total energy 1.0J is trapped in a potential given by the functions shown in the figure. Calculate the period of its oscillation:



Solution: The velocity of the particle as a function of position can be calculated from conservation of mechanical energy:

$$TotalEnergy = PE + KE = PE + \frac{1}{2}m\dot{x}^2 \rightarrow \dot{x} = \pm\sqrt{\frac{2(TE - PE)}{m}},$$

To get the period of oscillation we integrate the differential of time $dt = \frac{dx}{\dot{x}}$ from one turning point $x_1 = -1m$ to the other $x_2 = 1m$ and multiply by 2:

$$T(period) = 2\int_{x1}^{x2} \frac{dx}{\dot{x}}$$

Between x=-1m and x=0 the velocity is: $\dot{x} = \sqrt{\frac{2(1-0)}{2}} = 1m/s$ and between x=0 and x=1m it is $\dot{x} = \sqrt{\frac{2(1-0.75)}{2}} = 0.5m/s$, so the period is:

$$T = 2\int_{-1}^{0} \frac{dx}{1} + 2\int_{0}^{1} \frac{dx}{0.5} = \mathbf{6s}$$

Problem 4.-.- Consider a model of a potential given by: $U = ae^{-bx} + cx$ where *a*, *b* and *c* are all positive constants.

Find the equilibrium point x_o where the force on a particle of mass *m* is zero and calculate the angular frequency of small oscillations around that point.

Solution: To find the equilibrium point: $\frac{dU}{dx} = -abe^{-bx} + c = 0 \rightarrow x_o = \frac{1}{b}\ln\left(\frac{ab}{c}\right)$

And to find the frequency for small oscillations:

$$\left.\frac{d^2U}{dx^2}\right|_{x_o} = ab^2e^{-bx_o} = ab^2\frac{c}{ab} = bc \qquad \rightarrow \omega_o = \sqrt{\frac{\frac{d^2U}{dx^2}}{m}} = \sqrt{\frac{bc}{m}}$$