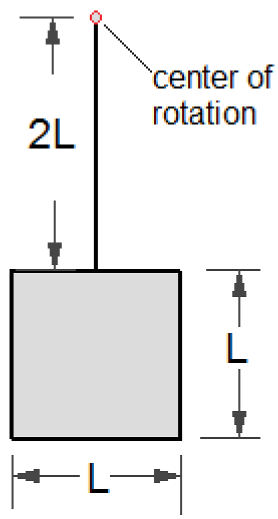
Classical Mechanics

Physical pendulum

Problem 1.- Calculate the period of the physical pendulum shown in the figure. It is made of a square of mass M and side L=0.1m and a rod of negligible mass and length 2L.

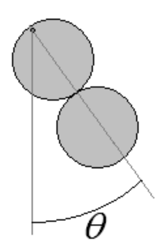
The moment of inertia of a square rotated about is center is $\frac{1}{6}ML^2$



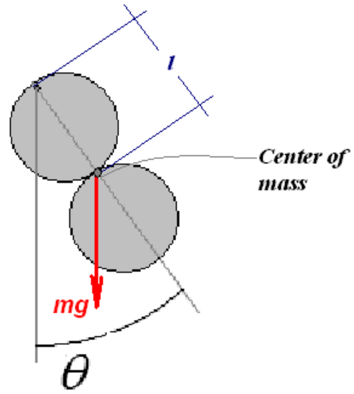
Solution: The moment of inertia is $I = \frac{1}{6}ML^2 + M(2.5L)^2 = \frac{77}{12}ML^2$ and then

$$T = 2\pi\sqrt{\frac{I}{Mg\ell}} = 2\pi\sqrt{\frac{\frac{77}{12}ML^2}{Mg\frac{5}{2}L}} = 2\pi\sqrt{\frac{77L}{30g}} = 1.02s$$

Problem 2.- Consider a pendulum made of two identical disks of radius R as shown in the figure. Calculate the period for small oscillations:



Solution: Consider the pendulum:



The torque around the center of rotation is given by:

$\tau = -Mgl\sin\theta$,

Where l is the distance between the center of mass and the center of rotation. The minus sign indicates that the torque is clockwise while the angle is measured counterclockwise.

The torque is equal to the derivative of the angular momentum with respect to time:

$$\tau = \frac{dL}{dt} = \frac{dI\omega}{dt} = \frac{Id\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

Therefore, the general equation for a physical pendulum is:

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

If the oscillations are small, we can approximate the angle to the sine of the angle, so:

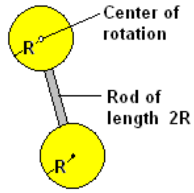
$I\frac{d^2\theta}{dt^2} \approx -mgl\theta$, which is the equation of a simple harmonic oscillator with angular frequency:

$\omega = \sqrt{\frac{mgl}{I}}$ and period: $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$

In the problem: $I = \frac{1}{2}MR^2 + MR^2 + \frac{1}{2}MR^2 + M(3R)^2 = 11MR^2$, where M is the mass of one disk, m=2M and $l = 2R$, so:

$$T = 2\pi\sqrt{\frac{11MR^2}{2Mg(2R)}} \rightarrow T = \pi\sqrt{\frac{11R}{g}}$$

Problem 3.- Consider the contraption shown in the figure. Calculate the period for small oscillations knowing that the mass of each disk is M and the mass of the rod that connects the two disks is also M.



Solution: We found in class that the period is: $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$

In the problem, we need to find the moments of inertia:

$$I_{disk-top} = \frac{1}{2}MR^2$$

$$I_{disk-bottom} = \frac{1}{2}MR^2 + M(4R)^2 = \frac{33}{2}MR^2$$

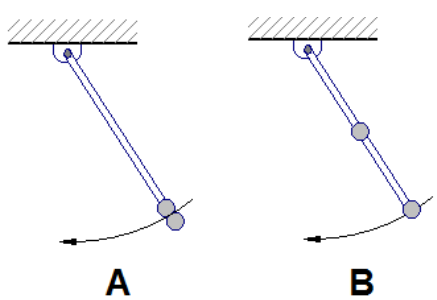
$$I_{rod} = \frac{ML^2}{12} + M(2R)^2 = \frac{M(2R)^2}{12} + M(2R)^2 = \frac{13MR^2}{3}$$

Therefore, the total moment of inertia is: $I = \frac{64}{3}MR^2$

In addition, by symmetry, the center of mass is located at a distance $l = 2R$ from the center of rotation, so:

$$T = 2\pi\sqrt{\frac{64MR^2}{3(3M)g(2R)}} \rightarrow T = \frac{8\pi}{3}\sqrt{\frac{2R}{g}}$$

Problem 4.- A long, straight, and massless rod pivots about one end in a vertical plane. In configuration A, shown in the figure, two small identical masses are attached to the free end; in configuration B, one mass is moved to the center of the rod. What is the ratio of the frequency of small oscillations of configuration B to that of configuration A?



Solution: The angular frequency of a "physical" pendulum is given by:

$\omega = \sqrt{\dfrac{mgl}{I}}$, where m is the mass, l is the distance from the center of mass to the center of rotation and I is the moment of inertia.

In the problem, we need to find the ratio of two angular frequencies, so we are looking for:

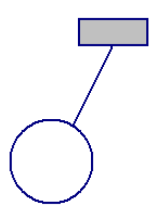
$$ratio = \frac{\omega_2}{\omega_1} = \frac{\sqrt{\dfrac{m_2 g l_2}{I_2}}}{\sqrt{\dfrac{m_1 g l_1}{I_1}}} = \frac{\sqrt{\dfrac{m_2 l_2}{I_2}}}{\sqrt{\dfrac{m_1 l_1}{I_1}}}$$

Case A: The mass is $2m$. The distance from the center of mass to the center of rotation is l and the moment of inertia is $2ml^2$

Case B: The mass is also $2m$. The distance from the center of mass to the center of rotation is $\dfrac{3l}{4}$ and the moment of inertia is $ml^2 + m\left(\dfrac{l}{2}\right)^2 = \dfrac{5ml^2}{4}$

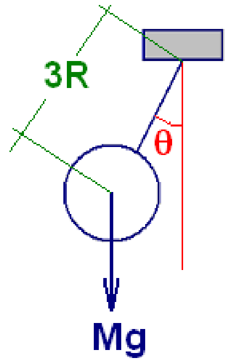
With these values, we get: $ratio = \dfrac{\sqrt{\dfrac{2m(3l/4)}{5ml^2/4}}}{\sqrt{\dfrac{2ml}{2ml^2}}} = \sqrt{\dfrac{6}{5}}$

Problem 5.- Calculate the period of small oscillations of a physical pendulum made of a hoop of radius 0.1m and mass M connected to a rod of negligible mass and length 0.2m.



Solution: The moment of inertia of the hoop around its center is MR^2 , but it is rotating around a point located at a distance 3R from its center, so the moment of inertia around that point is $I = MR^2 + M(3R)^2 = 10MR^2$

The center of mass of the hoop is at 3R from the center of rotation.



The equation of motion can be derived from Newton's second law for rotating objects:

$$\tau = I\alpha \rightarrow Mg(3R)\sin\theta = -(10MR^2)\ddot{\theta}$$

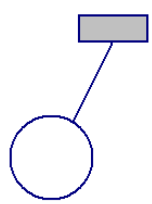
For small oscillations, we can approximate $\sin\theta \approx \theta$ and then the equation becomes:

$$Mg(3R)\theta = -(10MR^2)\ddot{\theta} \rightarrow \ddot{\theta} = -\frac{3g}{10R}\theta$$

The solution of this differential equation is a sine function with $\omega = \sqrt{\frac{3g}{10R}}$, so the period is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{10R}{3g}} = 2\pi\sqrt{\frac{10(0.1)}{3(9.8)}} = \mathbf{1.16s}$$

Problem 5a.- Calculate the period of small oscillations of a physical pendulum made of a solid sphere of radius 0.1m and mass M connected to a rod of mass M and length 0.2m.



Solution: The period is $T = 2\pi\sqrt{\frac{I}{mg\ell}}$, so we need to find the moment of inertia, total mass and the distance to the center of mass.

The moment of inertia:

- The moment of the sphere around its center is $2/5MR^2$, but it is rotating around a point located at a distance 0.3m from its center, so the moment of inertia around that point is $I = 2/5M\,0.1^2 + M(0.3)^2 = 0.094M$, this is called the Parallel Axis Theorem.
- The rod has a moment of inertia of $I = 1/3M\,0.2^2 = 0.0133M$

Therefore, the total moment of inertia is $I = 0.1073M$

Total mass:

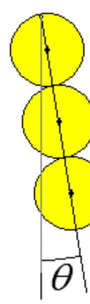
It is $2M$

Distance to the center of mass:

The center of mass of the sphere is 0.3m from the center of rotation and the center of mass of the rod is 0.1m from the center of rotation, so the center of mass of the system is 0.2m from the center of rotation (in this case it's just the middle point between the two centers of mass), so $\ell = 0.2$

The period is: $T = 2\pi\sqrt{\dfrac{0.1073M}{2M \times 9.8 \times 0.2}} =$ **1.04 s**

Problem 6.- Consider a pendulum made of three identical disks of radius R as shown in the figure. Calculate the radius of the disks if the period for small oscillations is **T=1.25 s**



Solution:

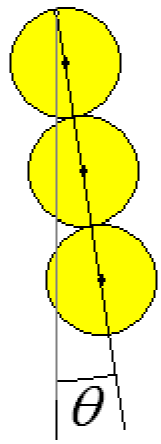
The period of the pendulum is $T = 2\pi\sqrt{\dfrac{I}{mg\ell}}$, but:

$$\ell = 3R$$
$$I = \frac{3}{2}MR^2 + M(R^2 + 9R^2 + 25R^2) = 36.5MR^2$$
$$m = 3M$$

So: $\mathrm{T} = 2\pi\sqrt{\dfrac{36.5\mathrm{MR}^2}{3\mathrm{Mg}(3\mathrm{R})}} = 2\pi\sqrt{\dfrac{36.5\mathrm{R}}{9\mathrm{g}}}$

Solving for R we get: $\mathrm{R} = \dfrac{9\mathrm{gT}^2}{36.5\times4\pi^2} = \dfrac{9\times9.8\times1.25^2}{36.5\times4\pi^2} = \mathbf{0.096\ m}$

Problem 6a.- Consider a pendulum made of three identical disks of radius R as shown in the figure. Calculate the period for small oscillations (an expression, with symbols) and the numerical value if R=0.2m.



Solution: The torque around the center of rotation is:

$\tau = -Mgl\sin\theta$,

Where l is the distance between the center of mass and the center of rotation. The minus sign indicates that the torque is clockwise while the angle is measured counterclockwise. And the torque is equal to the derivative of the angular momentum with respect to time:

$$\tau = \frac{dL}{dt} = \frac{dI\omega}{dt} = \frac{Id\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

Therefore, the general equation for a physical pendulum is:

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

If the oscillations are small, we can approximate the angle to the sine of the angle, so:

$I\dfrac{d^2\theta}{dt^2} \approx -mgl\theta$, which is the equation of a simple harmonic oscillator with angular frequency: $\omega = \sqrt{\dfrac{mgl}{I}}$ and period: $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{mgl}}$

In the problem:

$$I = \left(\frac{1}{2}MR^2 + MR^2\right) + \left(\frac{1}{2}MR^2 + M(3R)^2\right) + \left(\frac{1}{2}MR^2 + M(5R)^2\right) = 36.5MR^2,$$

Where M is the mass of one disk,

m=3M and $l = 3R$, so: $T = 2\pi\sqrt{\dfrac{36.5MR^2}{3Mg(3R)}} \rightarrow T = \pi\sqrt{\dfrac{146R}{9g}}$

If R=0.2: $T = \pi\sqrt{\dfrac{146\times 0.2}{9\times 9.8}} =$ **1.81 s**