Classical Mechanics

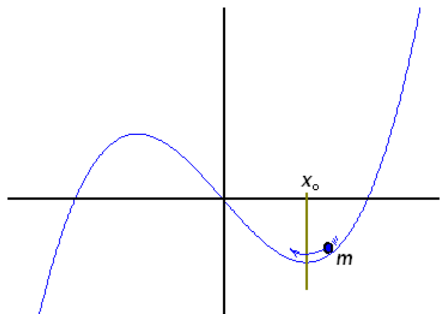
SHO

Problem 1.- The potential:

$$\mathrm{V} = ax^3 - bx$$

Where a and b are positive constants, has a minimum at $x_o = \sqrt{\frac{b}{3a}}$

A particle of mass m is trapped by the potential. Find the angular frequency of small oscillations around x_o



Solution: The potential given has a minimum when $x = x_o = \sqrt{\frac{b}{3a}}$, which means that it will be an equilibrium point, in other words the force will be zero there.

Expanding the potential around x_o in a Taylor series, we get:

$$\mathrm{V} = \mathrm{V}(x_o) + \left.\frac{d\mathrm{V}}{dx}\right|_{x_o}(x - x_o) + \frac{1}{2}\left.\frac{d^2\mathrm{V}}{dx^2}\right|_{x_o}(x - x_o)^2 + \ldots$$

The first term in the expansion is a constant and it will not produce a force because its derivative is zero. The second term in the expansion is zero since x_o is an equilibrium point.
If the third term is nonzero, it will dominate the series for small displacements since the next terms depend on the displacement cubed or higher powers.

Taking into account only the third term we notice it has the form of a spring potential energy ($\frac{1}{2}kx^2$), where $k = \left.\frac{d^2\mathrm{V}}{dx^2}\right|_{x_o}$, so the angular frequency is given by:

$$\omega_o = \sqrt{\frac{1}{m}\left.\frac{d^2\mathrm{V}}{dx^2}\right|_{x_o}}$$

In our problem: $\omega_o = \sqrt{\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x_o}} = \sqrt{\frac{6ax_o}{m}} = \sqrt{\frac{6a\sqrt{\frac{b}{3a}}}{m}} = \sqrt{\frac{2\sqrt{3ab}}{m}}$

Problem 2.- Consider a model of a potential given by: $U = ae^{-bx} + cx$ where a, b and c are all positive constants.
Find the equilibrium point x_o where the force on a particle of mass m is zero and calculate the angular frequency of small oscillations around that point.

Solution.- To find the equilibrium point: $\frac{dU}{dx} = -abe^{-bx} + c = 0 \rightarrow x_o = \frac{1}{b}\ln\left(\frac{ab}{c}\right)$
And to find the frequency for small oscillations:

$$\frac{d^2U}{dx^2}\bigg|_{x_o} = ab^2e^{-bx_o} = ab^2\frac{c}{ab} = bc \qquad \rightarrow \omega_o = \sqrt{\frac{\frac{d^2U}{dx^2}}{m}} = \sqrt{\frac{bc}{m}}$$

Problem 3.- The motion of the ground during certain earthquake can be modeled as a sinusoidal vertical motion of frequency f=2.5Hz. Calculate the maximum amplitude of the earthquake that does not cause loose objects to lose contact with the ground.

Solution: The vertical position can be described with the equation:
$x(t) = A\cos(\omega_o t - \delta)$, so the acceleration is $a = \frac{d^2x}{dt^2}x(t) = -A\omega_o^{\ 2}\cos(\omega_o t - \delta)$,
Notice that the maximum acceleration is $A\omega_o^{\ 2}$ and if this is equal to "g" the objects will lose contact with the ground.

So $A\omega_o^{\ 2} = g \rightarrow A = \frac{g}{\omega_o^{\ 2}} = \frac{g}{(2\pi f)^2} = \frac{9.8m/s^2}{(2\times 3.1416\times 2.5Hz)^2} =$ **0.039 m**

Problem 4.- A mass of 2 kg is attached to a spring with constant 18 N/m. It is then displaced to the point x = 2. How much time does it take for the block to travel to the point x = 1?

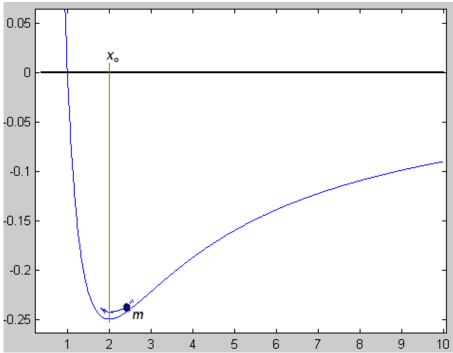
Solution: The value of ω_o is: $\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{18}{2}} = 3rad/s$
The amplitude is A=2, so the position is given by:

$x(t) = 2\cos(3t)$

When x=1 we have: $1 = 2\cos(3t) \rightarrow \cos(3t) = 0.5 \rightarrow 3t = \pi/3 \rightarrow t = \pi/9 \approx$ **0.35 s**

Problem 5.- The potential: $\quad V = \dfrac{1}{x^2} - \dfrac{1}{x}$

Has a minimum at $x_o = 2$, find the angular frequency of small oscillations around x_o for a particle of mass m=0.5



Solution: Using the Taylor expansion:

$$V = V(x_o) + \left.\frac{dV}{dx}\right|_{x_o}(x - x_o) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_o}(x - x_o)^2 + \ldots$$

We notice that $\left.\dfrac{d^2V}{dx^2}\right|_{x_o} = \left.\dfrac{6}{x^4} - \dfrac{2}{x^3}\right|_{x_o} = \dfrac{6}{2^4} - \dfrac{2}{2^3} = \dfrac{1}{8}$

Since it is positive, it will dominate small oscillations and the motion will be approximately simple harmonic with angular frequency:

$$\omega_o = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_o}}{m}} = \sqrt{\frac{1/8}{0.5}} = \textbf{0.5 rad/s}$$

Problem 5a.- The potential: $\quad V = \dfrac{1}{x^{12}} - \dfrac{1}{x^6}$

Has a minimum at x_o, find the angular frequency of small oscillations around x_o for a particle of mass m=1

Solution: $\dfrac{dV}{dx} = -\dfrac{12}{x^{13}} + \dfrac{6}{x^7} = 0 \rightarrow x_o = \sqrt[6]{2} = 1.122$

And $\quad \dfrac{d^2V}{dx^2} = \dfrac{156}{x^{14}} - \dfrac{42}{x^8} = \dfrac{156}{\sqrt[6]{2}^{14}} - \dfrac{42}{\sqrt[6]{2}^{8}} = 14.29$ then $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{14.29}{1}} = 3.8 rad/s$

Problem 5b.- Consider the potential $V = \frac{1}{x^4} - \frac{1}{x^2}$

Locate the minimum of this potential and find the angular frequency of small oscillations around the minimum for a particle of mass m=7.1kg

Solution: Find the minimum by $\frac{dV}{dx} = -\frac{4}{x^5} + \frac{2}{x^3} = 0 \rightarrow x_o = \sqrt{2}$

Then we identify $k = \left.\frac{d^2V}{dx^2}\right|_{x_o} = \left.\frac{20}{x^6} - \frac{6}{x^4}\right|_{x_o} = \frac{20}{8} - \frac{6}{4} = 1$

Since it is positive it will dominate small oscillations and the motion will be approximately simple harmonic with angular frequency:

$$\omega_o = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x_o}}{m}} = \sqrt{\frac{1}{7.1}} = \textbf{0.38 rad/s}$$