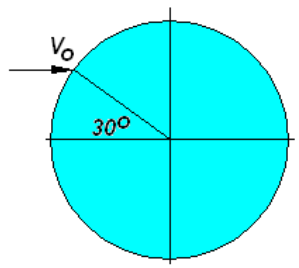
Classical Mechanics

SHO in 2D

Problem 1.- Consider a particle that has mass *m*, but no charge and very weak interactions with matter except for gravitation.
That particle enters our planet as shown in the figure at time t=0 with velocity:

$V_o = 3{,}950$m/s.

Find the position vector (x, y) as a function of time while the particle is inside the planet.
Assume the Earth has constant density.



Solution: Inside the planet, the gravitational force is proportional to the distance to the center, similar to a 2-dimensional simple harmonic oscillator. The angular frequency in the X and Y directions will be the same.
The gravitational force is:

$$F = \frac{GM'm}{r^2}$$

Where *M'* is the mass inside the sphere of radius *r*.

We assume the density is constant, so: $M' = \rho\frac{4}{3}\pi r^3 = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$, where *M* is the total mass of the planet.

Back in the force equation: $F = \frac{GM'm}{r^2} = \frac{GMm}{R^3}r$

This implies a 2D oscillator with $\omega_o = \sqrt{\frac{GM}{R^3}}$

In our case: $\omega_o = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{6.67\times10^{-11}\times5.98\times10^{24}}{(6.38\times10^6)^3}}$ =0.001239 rad/s

The solution for such a problem is:

$$x = A_x\cos(\omega_o t - \delta_x) \qquad y = A_y\cos(\omega_o t - \delta_y)$$

$$v_x = A_x\omega_o\sin(\omega_o t - \delta_x) \qquad v_y = A_y\omega_o\sin(\omega_o t - \delta_y)$$

The initial conditions are:

$$x(0) = -R\cos 30^o \qquad y(0) = R\sin 30^o$$

$$v_x(0) = 3{,}950 m/s \qquad v_y(0) = 0$$

From the initial conditions for y we get that $\delta_y = 0$ and $A_y = R\sin 30^o = R/2$

The initial conditions for x are a bit trickier, but you just need to realize that:

$(x(0))^2 + \frac{(v_x(0))^2}{\omega_o^{\,2}} = A_x^{\,2}$, and that allows you to find the amplitude.

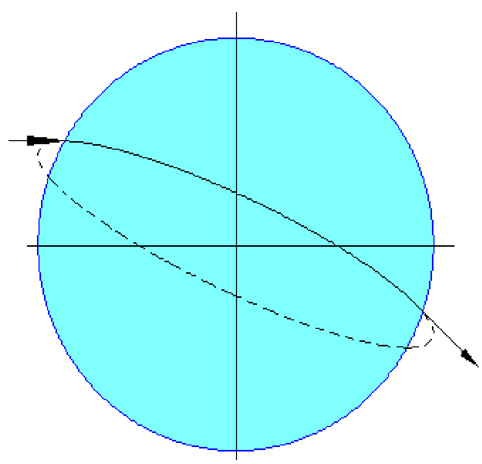
$$A_x = \sqrt{(x(0))^2 + \frac{(v_x(0))^2}{\omega_o^{\,2}}} = \sqrt{(6.38\times 10^6 \times \cos 30^o)^2 + \frac{3950^2}{0.001239^2}} = 6.38\times 10^6 m$$

This amplitude is precisely R, so:

$$x(0) = -R\cos 30^o = R\cos(-\delta_x) \rightarrow \delta_x = 150^o$$

The solution is then: $\quad x = R\cos(0.001239t - 150^o) \qquad y = \frac{R}{2}\cos(0.001239t)$

The trajectory is shown in the figure:



Inside the planet, the trajectory is an ellipse, as expected, with the center of the ellipse being the center of the planet. Outside the planet the trajectory is also an ellipse because the total energy is negative (this is a bound state). However, this ellipse will have a *focus* (not its center) located at the center of the planet.