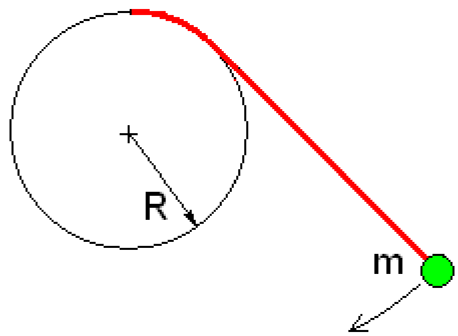
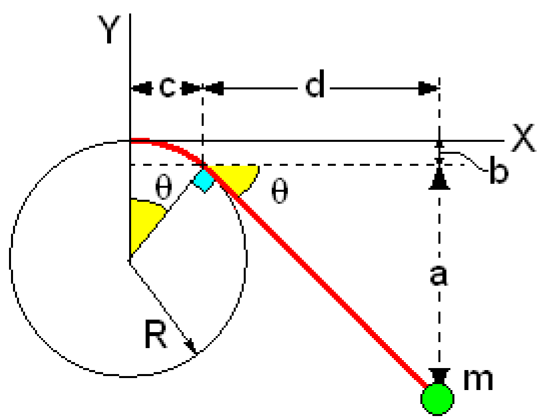
Classical Mechanics

Small Oscillations

Problem 1.- Consider a pendulum made by attaching a mass m to a string with negligible mass and length l and fixing one end to the highest point of a cylinder of radius R and letting the mass hung to the side. Find the equation of motion of the mass and the frequency for small oscillations.



Solution: First we write x and y in terms of the variable



$$x = c + d = R\sin\theta + (l - R\theta)\cos\theta$$
$$y = -a - b = -R(1-\cos\theta) - (l - R\theta)\sin\theta$$

Notice that x and y depend on θ only, so that is our only coordinate. We will find just one equation of motion. Next, we calculate the velocities:

$$\dot{x} = \dot{\theta}\left[R\cos\theta + (l - R\theta)\sin\theta - R\cos\theta\right] = \dot{\theta}(l - R\theta)\sin\theta$$
$$\dot{y} = \dot{\theta}\left[-R\sin\theta - (l - R\theta)\cos\theta + R\sin\theta\right] = -\dot{\theta}(l - R\theta)\cos\theta$$

These equations allow us to write down the kinetic energy:

$$K.E. = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{\theta}^2(l - R\theta)^2$$

To find the Lagrangian, we also need the potential energy:

$$P.E. = mgy = mg[-R(1-\cos\theta)-(l-R\theta)\sin\theta]$$

With these two equations the Lagrangian is:

$$L = K.E. - P.E. = \frac{1}{2}m\dot{\theta}^2(l-R\theta)^2 + mg[R(1-\cos\theta)+(l-R\theta)\sin\theta]$$

To find the equation of motion we take derivatives.
Partial derivative of L with respect to $\dot{\theta}$:

$$\frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}(l-R\theta)^2$$

Then the total derivative of this last expression with respect to time:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ddot{\theta}(l-R\theta)^2 - 2mR\dot{\theta}^2(l-R\theta)$$

And the derivative of L with respect to θ:

$$\frac{\partial L}{\partial \theta} = -Rm\dot{\theta}^2(l-R\theta) + mg(l-R\theta)\cos\theta$$

Then the equation of motion is:

$$m\ddot{\theta}(l-R\theta)^2 - 2mR\dot{\theta}^2(l-R\theta) = -Rm\dot{\theta}^2(l-R\theta) + mg(l-R\theta)\cos\theta$$

$$\ddot{\theta}(l-R\theta) - R\dot{\theta}^2 - g\cos\theta = 0$$

Change of variables to simplify equation:

$\theta = \frac{\pi}{2} + \delta$ $\qquad$ $\cos\theta = -\sin\delta$ $\qquad$ $l - R\frac{\pi}{2} = C \rightarrow l - R\theta = C - R\delta$

The equation with the changes looks as follows: $\ddot{\delta}(C-R\delta) - R\dot{\delta}^2 + g\sin\delta = 0$

Small oscillations imply that δ is very small, so we can ignore second order terms and approximate $\sin\delta \approx \delta$, so:

$C\ddot{\delta} + g\delta = 0$ whose frequency of oscillation is $\omega = \sqrt{\frac{g}{C}} = \sqrt{\frac{g}{l - \frac{\pi}{2}R}}$

For very small oscillations, the frequency is the same equation as for a normal pendulum, but the length is only the part that hangs on the side in equilibrium.