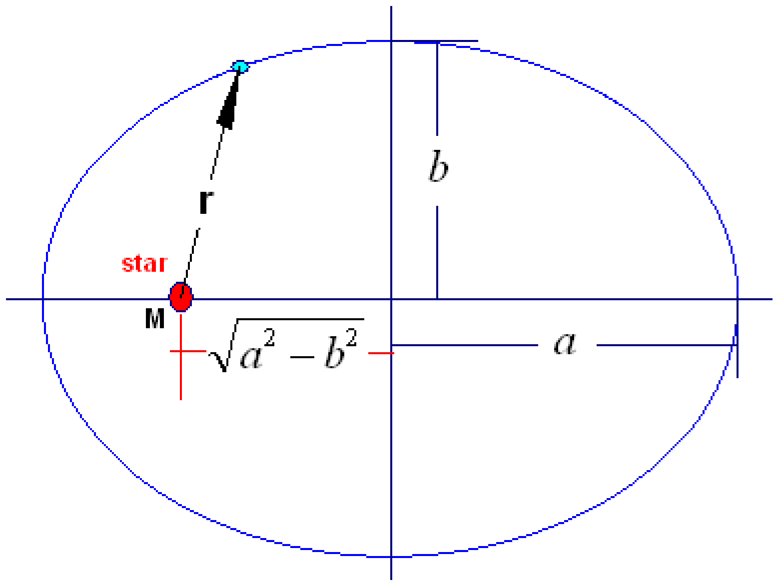
Classical Mechanics

Orbits

Problem 1.- A small planet of mass *m* is in elliptical orbit around a star of mass *M*. The trajectory can be described by the parametric equations:

$x = a\cos(\phi)$ and $y = b\sin(\phi)$

With the star located at $(-\sqrt{a^2 - b^2}, 0)$ and the period of the orbit being T.



a) Calculate the mass "M" of the star as a function of the other parameters given above. Assume $\mu \approx m$.

b) Calculate the maximum speed of the planet.

c) Calculate the minimum speed of the planet.

Solution:

a) To calculate the mass of the star notice that the centripetal force has to be equal to the gravitational force, which for a circular orbit of radius R means:

$$m\omega^2 R = G\frac{mM}{R^2} \rightarrow m\frac{4\pi^2}{T^2}R = G\frac{mM}{R^2}$$
$$\rightarrow M = \frac{4\pi^2 R^3}{GT^2}$$

But in the case of an elliptical orbit we showed in class that the value of R should be substituted by the semi-major axis "*a*", so the answer is:

$$M = \frac{4\pi^2 a^3}{GT^2}$$

b) The maximum speed of the planet should happen when the planet is the closest to the star, which corresponds to a distance $r = a - \sqrt{a^2 - b^2}$, recall that the total energy of the planet is

$-G\dfrac{mM}{2a}$.

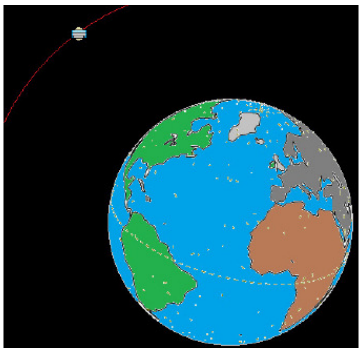
Writing the equation of conservation of mechanical energy:

$$-G\frac{mM}{2a} = -G\frac{mM}{a - \sqrt{a^2 - b^2}} + \frac{1}{2}mv^2 \rightarrow v_{\max} = \sqrt{\frac{GM}{a}\left[\frac{1+\sqrt{1-b^2/a^2}}{1-\sqrt{1-b^2/a^2}}\right]}$$

c) In a similar way you can calculate the minimum speed, which will happen when $r = a + \sqrt{a^2 - b^2}$, so:

$$v_{\min} = \sqrt{\frac{GM}{a}\left[\frac{1-\sqrt{1-b^2/a^2}}{1+\sqrt{1-b^2/a^2}}\right]}$$

Problem 2.- Calculate the period of a satellite whose altitude over the surface of our planet is equal to one Earth radius.



Solution: The centripetal force is: $ma = \frac{mg}{4} \rightarrow a = \frac{g}{4} = 2.45 m/s^2$

Then we can write the following equation:

$$a = \frac{g}{4} = \omega^2 r = \frac{4\pi^2}{T^2} r \rightarrow T = \sqrt{\frac{4\pi^2}{a} r} = 2\pi\sqrt{\frac{2R}{a}}$$

$$T = 2\pi\sqrt{\frac{2\times 6.38\times 10^6}{2.45}} = \mathbf{14{,}300\ s}$$

Problem 3.- A particle of mass M moves in a circular orbit of radius r around a fixed point under the influence of an attractive force $F = \frac{K}{r^3}$, where K is a constant. If the potential energy of the particle is zero at an infinite distance from the force center, the total energy of the particle in the circular orbit is

Solution: You can calculate the potential energy as follows:

$$P.E. = \int_{\infty}^{r} \frac{K}{r^3} dr = -\frac{K}{2r^2}$$

Now, the kinetic energy can be calculated by considering the centripetal force:

$$ma = m\frac{v^2}{r} = \frac{K}{r^3} \rightarrow KE = m\frac{v^2}{2} = \frac{K}{2r^2}$$

Therefore, the total energy is zero.

Problem 3.- The minimum speed of a particle to escape the solar system from the surface of the Sun is 618,000 m/s. Calculate the speed of such a particle when it reaches the orbit of the Earth, which is 1.5×10^{11}m from the sun. The mass of the sun is 1.99×10^{30} kg and its radius is 6.96×10^{8} m.

Solution: The total energy of the particle is zero, so when it reaches our orbit we have:

$$-G\frac{m_{particle} m_{sun}}{r_{orbit}} + \frac{1}{2} m_{particle} v^2 = 0 \rightarrow G\frac{m_{sun}}{r_{orbit}} = \frac{1}{2} v^2$$

$$\rightarrow v = \sqrt{\frac{2Gm_{sun}}{r_{orbit}}} = \sqrt{\frac{2(6.67\times 10^{-11})(1.99\times 10^{30})}{1.5\times 10^{11}}} = \mathbf{42.1\ km/s}$$

Problem 4.- A satellite is in circular orbit around the Earth when it collides with a piece of space junk and its velocity changes from $\vec{v}_o$ just before the collision to $1.5\vec{v}_o$ just after. What kind of orbit will it be now (circular, elliptical, parabolic, etc)?

Solution: increasing the speed 50% will increase the kinetic energy 125%, so the total energy will be positive and the trajectory will be a hyperbola.

Problem 5.- A satellite of mass m is in circular orbit of radius R. Another satellite of mass $2m$ is in a circular orbit of radius $7R$. Calculate the ratio of the gravitational forces acting on the two satellites $\frac{F_{1st-satellite}}{F_{2nd-satellite}}$

Solution: $\frac{F_{1st-satellite}}{F_{2nd-satellite}} = \frac{G\frac{Mm}{R^2}}{G\frac{M\,2m}{(7R)^2}} = 24.5$

Problem 6.- A satellite of mass m=84 kg is in circular orbit around the Earth at a distance of 750 km above the surface. Calculate its kinetic energy and its speed.

Solution: The speed of a satellite in circular orbit is $\sqrt{\frac{GM}{r}}$. In the present case $M = 5.98\times10^{24}$, $r = 6.38\times10^{6} + 0.75\times10^{6} = 7.13\times10^{6}$, so:

$$v = \sqrt{\frac{5.98\times10^{24}\times6.67\times10^{-11}}{7.13\times10^{6}}} = \mathbf{7{,}480\ m/s}$$

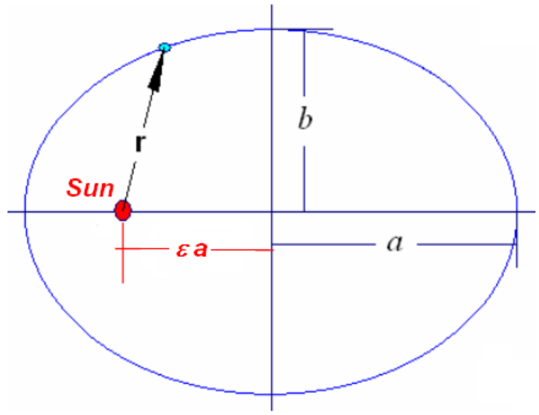
The kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}(84)(7480)^2 = $ **2.35 GJ**

Problem 7.- Eros is an asteroid of the solar system. Its period is 1.7610 years and its eccentricity is 0.2230 Calculate its closest distance to the Sun in terms of the orbital radius of the Earth. Should we worry about a collision in the future?

Solution: We can use Kepler's third law to find the semi major axis (*a*):

$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3}$, where "1" stands for Eros and "2" for the Earth, so:

$$\frac{(1.7610\,year)^2}{a_1^3} = \frac{(1\,year)^2}{R_{OrbitEarth}^3} \rightarrow a_1 = \sqrt[3]{1.7610^2}\,R_{OrbitEarth} = 1.4583R_{OrbitEarth}$$



Notice in the figure that the shortest distance between the Sun and Eros is: $r_{\min} = a - \varepsilon a$, and given the values of the problem this distance is:

$$r_{\min} = a - \varepsilon a = a - 0.2230a = 0.777a = 0.777(1.4583 R_{OrbitEarth}) = 1.1331 R_{OrbitEarth}$$

Since this distance is larger than the radius of the Earth's orbit, we do not need to worry.

Problem 7a.- A currently unnamed asteroid of the solar system has a period of 1.73 years and its eccentricity is 0.23
Calculate its closest distance to the Sun in terms of the orbital radius of the Earth. Should we worry about a collision in the future?

Solution: First, we calculate the semi major axis of the asteroid's orbit:

$$\frac{(1\,year)^2}{1A.U.^3} = \frac{(1.73\,year)^2}{a^3} \rightarrow a = (1.73)^{2/3} A.U. = 1.441 A.U.$$

The minimum distance to the sun is:

$$d_{\text{minimum}} = a(1-\varepsilon) = 1.441 \times (1 - 0.23) = 1.11 \text{ A.U.}$$

Since this is larger than one astronomical unit, we are safe.

Problem 8.- A particle of mass m is attracted towards the origin of coordinates with a force F=kr, where k is a constant. If the trajectory of the particle is a circle, calculate its kinetic energy, knowing that the radius of the circle is R.

Solution: We know that the force has to be equal to the centripetal force, so:

$$\mathrm{F} = \mathrm{kR} = \mathrm{m}\frac{\mathrm{v}^2}{\mathrm{R}} \rightarrow \mathrm{K.E.} = \frac{1}{2}\mathrm{mv}^2 = \frac{1}{2}\mathrm{kR}^2$$

You can also find the same result using the Virial Theorem.

Problem 9.- Knowing that the period of an Earth satellite orbiting at sea level would be 80 minutes (if not for the atmosphere). In terms of Earth's radius R_e, the radius of a synchronous satellite orbit (period 24 hours) is:

(A) $3R_e$ (B) $7R_e$ (C) $18R_e$ (D) $320R_e$ (E) $5800R_e$

Solution: Using Kepler's third law we get:

$$\frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3} \rightarrow R_2 = R_1 \sqrt[3]{\frac{T_2^2}{T_1^2}} = R_e \sqrt[3]{\frac{(24 \times 60\,\text{min})^2}{(80\,\text{min})^2}} = R_e \sqrt[3]{18^2} \approx 7R_e$$

Answer: **B**