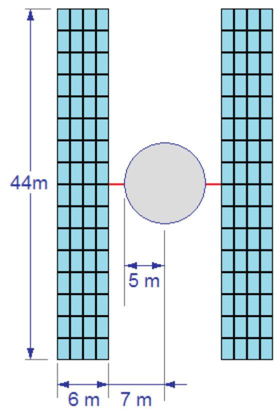
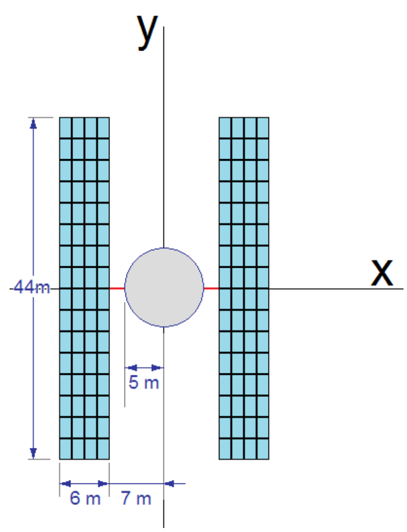
Classical Mechanics

Moment of Inertia

Problem 1.- A mini space station can be modeled as a hollow sphere of mass 24,000 kg and two rectangular solar panels of mass 1,200 kg each as shown in the figure. Calculate the moment of inertia tensor.



Solution: We define the x, y and z axis as shown in the figure below. First, we realize that due to symmetry the off-diagonal elements in the moment of inertia tensor are all zero, so we need to concentrate on I_{xx}, I_{yy} *and* I_{zz}



Sphere: Notice that for the hollow sphere $\int (x^2+y^2+z^2)dm = MR^2$, and due to symmetry $\int x^2 dm = \int y^2 dm = \int z^2 dm$, so the contribution to I_{xx}, I_{yy} *and* I_{zz} will be $\frac{2}{3}MR^2$ or $\frac{2}{3}(24{,}000)(5)^2 = 400{,}000 kgm^2$

Panels: the value of $\int z^2 dm$ is zero because the panels are thin, so it just remains finding $\int x^2 dm$ and $\int y^2 dm$, the latter is easier because it is the same as the moment of inertia of a rod rotated about its center $\int y^2 dm = \frac{1}{12} ML^2 = \frac{1}{12}(1{,}200)44^2 = 193{,}600 kgm^2$ for each panel, so $I_{xx-panels} = 2 \times 193{,}600 kgm^2 = 386{,}600 kgm^2$.

More interesting is finding $\int x^2 dm$. We can still use the formula $\frac{1}{12} ML^2$ for a rotation about their center of the panel, but the center of the panel is not the center of rotation, so we need to add $M\ell^2$ where ℓ is the distance between the parallel axes:

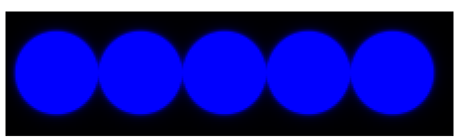
$\int x^2 dm = \frac{1}{12} ML^2 + M\ell^2 = \frac{1}{12}(1{,}200)(6)^2 + 1200(10)^2 = 123{,}600 kgm^2$ for each panel, so $I_{yy-panels} = 2 \times 123{,}600 kgm^2 = 247{,}200 kgm^2$

Finally in the z-direction we have $I_{xx-panels} = 247{,}200 kgm^2 + 386{,}600 kgm^2$

The tensor is then

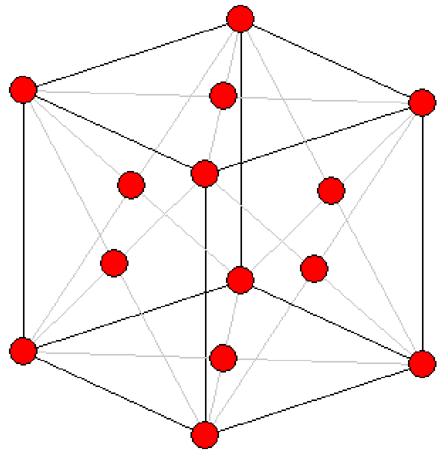
$$\vec{I} = \begin{bmatrix} 787{,}200 kgm^2 & 0 & 0 \\ 0 & 647{,}200 kgm^2 & 0 \\ 0 & 0 & 1{,}034{,}400 kgm^2 \end{bmatrix}$$

Problem 2.- Five identical disks have mass m and radius R each. They are glued together as shown in the figure. Calculate the moment of inertia of the group about an axis of rotation perpendicular to the plane and through the center of the middle disk.



Solution: $I = 5 \times \frac{1}{2} MR^2 + 2 \times M(2R)^2 + 2 \times M(4R)^2 = 42.5 MR^2$

Problem 3.- Calculate the moment of inertia tensor of the following object made with 1kg masses located at the corners of a cube of side L and at the centers of each face. Use one of the corners as the origin of coordinates.



Solution: Due to symmetry, we only need to calculate two summations:

$$\sum mx^2 = \sum my^2 = \sum mz^2 = 4\times\left(\frac{L}{2}\right)^2 + 5\times L^2 = 6L^2 \text{ and}$$

$$\sum mxy = \sum mxz = \sum myz = 2\times\left(\frac{L}{2}\right)^2 + 2\frac{L}{2}L + 2\times L^2 = 3.5L^2$$

The matrix is

$$\begin{bmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{bmatrix} = L^2 \begin{bmatrix} 12 & -3.5 & -3.5 \\ -3.5 & 12 & -3.5 \\ -3.5 & -3.5 & 12 \end{bmatrix}$$

Problem 3a.- A cluster is made of 9 identical atoms of mass **m** each, that are located at the corners of a cube of side **L** and one in the center of the cube. Calculate the moment of inertia tensor assuming that the origin of coordinates is at one corner and the cube is in the positive octet (x>0, y>0, z>0)

Solution: Due to symmetry, we only need to find one diagonal element and one off diagonal element:

$$I_{xx} = \sum m_i (y_i^{\,2} + x_i^{\,2}) = 8.5mL^2$$
$$I_{xy} = -\sum m_i (y_i x_i) = -2.25mL^2$$

Therefore, the matrix is: $\vec{I} = mL^2 \begin{bmatrix} 8.5 & -2.25 & -2.25 \\ -2.25 & 8.5 & -2.25 \\ -2.25 & -2.25 & 8.5 \end{bmatrix}$

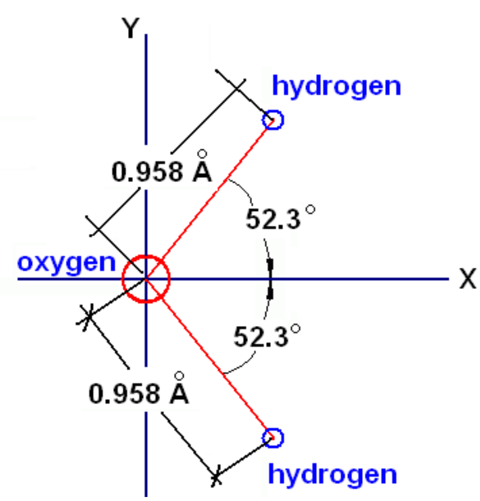
Problem 3b.- A cluster is made of 8 atoms of mass **m** each, that are located at the corners of a cube of side **L**. Calculate the moment of inertia tensor assuming that the origin of coordinates is at one corner and the cube is in the positive octet (x>0, y>0, z>0)

Solution: Due to symmetry, we only need to find one diagonal element and one off diagonal element:

$$I_{xx} = \sum m_i (y_i^{\ 2} + x_i^{\ 2}) = 8mL^2$$
$$I_{xy} = -\sum m_i (y_i x_i) = -2mL^2$$

Therefore, the matrix is: $\vec{I} = mL^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix}$

Problem 4.- Calculate the principal moments of inertia of a water molecule.



Solution:

$$I_{xx} = \sum m\left(y^2 + z^2\right) = 1u\left((0.958\,\text{Å} \sin 52.3^\circ)^2 + 0^2 \right) + 1u\left((-0.958\,\text{Å} \sin 52.3^\circ)^2 + 0^2 \right) = 1.149\,\text{Å}^2 u$$

$$I_{yy} = \sum m\left(x^2 + z^2\right) = 1u\left((0.958\,\text{Å} \cos 52.3^\circ)^2 + 0^2 \right) + 1u\left((0.958\,\text{Å} \cos 52.3^\circ)^2 + 0^2 \right) = 0.686\,\text{Å}^2 u$$

$$I_{zz} = \sum m\left(x^2 + y^2\right) = 1.8355\,\text{Å}^2 u$$

Problem 5.- A homogeneous disk of mass M_1 and radius R_1 is attached to a larger disk of radius R_2 and mass M_2. Both disks roll down an incline plane as shown in the figure, with the smaller disk in contact with the plane surface without slipping.
Calculate the linear acceleration of the center of mass of the disks.