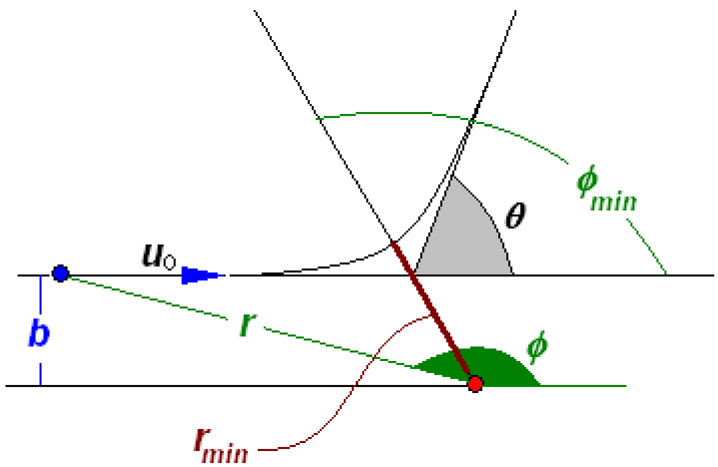
Classical Mechanics

Cross section

Problem 1: Find the differential cross section for particles of mass m and initial velocity u_o scattered by a fixed force center that follows the law: $F = \frac{k}{r^3}$

Solution:
First we try to find a relationship between the impact parameter b and the angle of scattering θ as follows:



Using polar coordinates:

$$x = r\cos\phi \quad y = r\sin\phi \rightarrow \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \quad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

Conservation of angular momentum tells us that:

$$\vec{L} = \vec{r}x\vec{p} = m(r\cos\phi, r\sin\phi)\times(\dot{r}\cos\phi - r\dot{\phi}\sin\phi, \dot{r}\sin\phi + r\dot{\phi}\cos\phi) =$$
$$mr\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - mr\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) = mr^2\dot{\phi} = -mbu_o$$

This gives us the relation: $r^2\dot{\phi} = -bu_o$ ….. Equation 1

Conservation of total energy tells us that:

$$P.E. + K.E. = \text{constant} \rightarrow \frac{1}{2}mu_o^2 = \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2) + \frac{k}{2r^2} \quad \text{…. Equation 2}$$

Replacing equation 1 in 2 we get:

$$\frac{1}{2}mu_o^2 = \frac{1}{2}m(\frac{b^2u_o^{\ 2}}{r^2} + \dot{r}^2) + \frac{k}{2r^2} \rightarrow \dot{r} = \sqrt{u_o^2 - \frac{k/m + b^2u_o^{\ 2}}{r^2}}$$

Notice that the minimum value of r happens when the derivative of r is zero, so:

$$\dot{r} = 0 = \sqrt{u_o^2 - \frac{k/m + b^2 u_o{}^2}{r_{\min}{}^2}} \rightarrow u_o^2 - \frac{k/m + b^2 u_o{}^2}{r_{\min}{}^2} = 0 \rightarrow r_{\min} = \sqrt{b^2 + \frac{k}{mu_o^2}}$$

Notice also that the equation $r^2\dot{\phi} = -bu_o$ can be written as:

$$r^2 \frac{d\phi}{dt} = r^2 \frac{d\phi}{dt}\frac{dr}{dr} = r^2 \frac{d\phi}{dr}\frac{dr}{dt} = r^2 \frac{d\phi}{dr}\dot{r} = -bu_o \rightarrow d\phi = -\frac{bu_o}{r^2\dot{r}}dr = -\frac{bu_o dr}{r^2\sqrt{u_o^2 - \frac{k/m + b^2 u_o{}^2}{r^2}}}$$

Changing variable: $z = \frac{1}{r} \rightarrow dz = -\frac{dr}{r^2}$

$$d\phi = \frac{bdz}{\sqrt{1 - r^2{}_{\min} z^2}} \rightarrow \pi - \phi_{\min} = \int_0^{z\min} \frac{bdz}{\sqrt{1 - r^2{}_{\min} z^2}} = \frac{b\sin^{-1}(zr_{\min})_0^{z\min}}{r_{\min}} = \frac{b\pi/2}{r_{\min}}$$

$$\phi_{\min} = \pi - \frac{b\pi/2}{r_{\min}} \rightarrow \theta = \pi - 2(\pi - \phi) = \pi - \frac{b\pi}{r_{\min}} = \pi - \frac{\pi}{\sqrt{1 + \frac{k}{mb^2u_o^2}}}$$

$$\theta = \pi - \frac{\pi}{\sqrt{1 + \frac{k}{mb^2u_o^2}}} \rightarrow b = \frac{\frac{1}{u_o}\sqrt{\frac{k}{m}}}{\sqrt{\left(\frac{\pi}{\pi - \theta}\right)^2 - 1}} \rightarrow db = \frac{\frac{1}{u_o}\sqrt{\frac{k}{m}}}{\left[\sqrt{\left(\frac{\pi}{\pi - \theta}\right)^2 - 1}\right]^3}\frac{\pi^2 d\theta}{(\pi - \theta)^3}$$

To get the differential cross section:

$$\sigma = \frac{dN}{d\Omega}\frac{A}{N} = \frac{2\pi bdbN/A}{2\pi\sin\theta d\theta}\frac{A}{N} = \frac{bdb}{\sin\theta d\theta} = \frac{\frac{1}{u_o}\sqrt{\frac{k}{m}}}{\sin\theta\sqrt{\left(\frac{\pi}{\pi - \theta}\right)^2 - 1}}\frac{\frac{1}{u_o}\sqrt{\frac{k}{m}}\frac{\pi^2}{(\pi - \theta)^3}}{\left[\sqrt{\left(\frac{\pi}{\pi - \theta}\right)^2 - 1}\right]^3}$$

$$\sigma = \frac{\pi^2 k}{mu_o^2\theta^2\sin\theta}\frac{\pi - \theta}{(2\pi - \theta)^2}$$

Problem 2.- A beam of sodium clusters of radius 3.5nm goes through a scattering cell filled with molecules of radius 0.23nm. Assuming hard sphere collisions, what is the density per area of these molecules (in $\frac{number}{m^2}$) so 1% of the beam is scattered?

Solution: The scattering cross-section for hard spheres with radii $r_1 = 3.5nm$ and $r_2 = 0.23nm$ is given by:

$$\sigma = \pi(r_1 + r_2)^2$$

If 1% of the beam is scattered it means that in average 1% of the surface is covered with the scattering molecules. Since 1% is very small we do not need to worry about a molecule eclipsing another and we will just assume that 1% of the area is covered with the molecules:

$$\frac{\#of\ molecules}{1m^2} = \frac{0.01}{\sigma} = \frac{0.01}{\pi(r_1+r_2)^2} = \frac{0.01}{\pi(3.5\times10^{-9}m + 0.23\times10^{-9}m)^2} = \mathbf{2.28\times10^{14}/m^2}$$

Problem 3.- Find the cross section of a gold nucleus that deflects alpha particles by more than 90 degrees. You can use the approximate impact parameter: $\mathrm{b} = \frac{\mathrm{Z_1 Z_2 e^2}}{8\pi\varepsilon_o \mathrm{E}}\cot\left(\frac{\Omega}{2}\right)$

Solution: The cross section will be:

$$\sigma = \pi b^2 = \pi\left[\frac{Z_1 Z_2 e^2}{8\pi\varepsilon_o E}\cot\left(\frac{\Omega}{2}\right)\right]^2$$

With the values:

$$Z_1 = 2$$

$$Z_1 = 79$$

$$e = 1.6\times10^{-19}\,coulomb$$

$$\cot\left(\frac{\Omega}{2}\right) = \cot\left(\frac{90^\circ}{2}\right) = 1$$

$$\frac{1}{4\pi\varepsilon_o} = 9\times10^9\,\frac{Nm^2}{C^2}$$

$$\sigma = \pi\left[\frac{Z_1 Z_2 e^2}{8\pi\varepsilon_o E}\cot\left(\frac{\Omega}{2}\right)\right]^2 = \frac{1.04\times10^{-51}}{E^2}$$