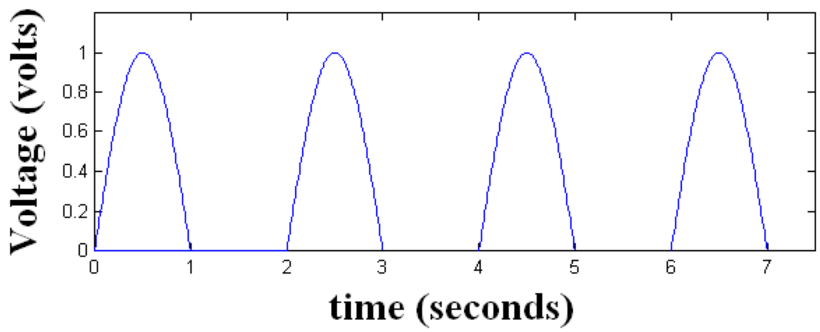
Classical Mechanics

Fourier series

Problem 1.- Obtain the first three terms of the Fourier series representation of the output of the half-wave rectifier (a_o, a_1 and b_1):



Solution: The function has a period of T=2s, so it can be written as:

$$f(t) = \frac{a_o}{2} + \sum a_n \cos\frac{2\pi nt}{2} + \sum b_n \sin\frac{2\pi nt}{2}$$

In the interval [0,2] the function is equal to: $f(t) = \begin{cases} \sin \pi t & if \quad 0 < t < \pi \\ 0 & otherwise \end{cases}$

Let's find the first terms:

$$a_o = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{2}\int_0^1 \sin \pi t dt = -\frac{\cos \pi t}{\pi}\bigg|_0^1 = \frac{2}{\pi}$$

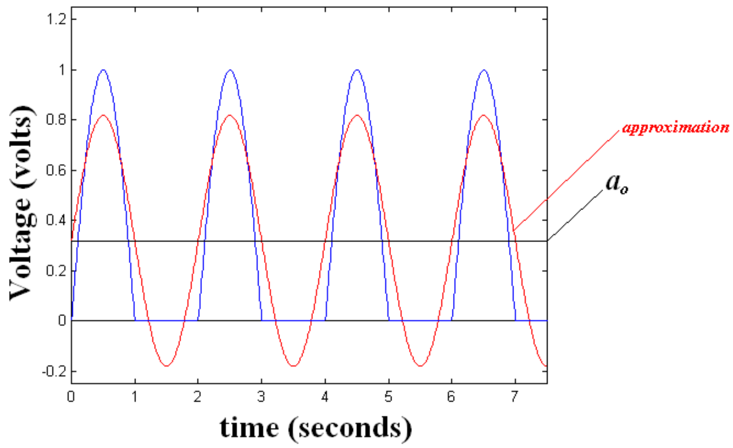
$$a_1 = \frac{2}{T}\int_0^T (\cos \pi t) f(t)dt = \frac{2}{2}\int_0^1 \cos \pi t \sin \pi t dt = \int_0^1 \frac{\sin 2\pi t}{2} dt = -\frac{\cos 2\pi t}{4\pi}\bigg|_0^1 = 0$$

$$b_1 = \frac{2}{T}\int_0^T (\sin \pi t) f(t)dt = \frac{2}{2}\int_0^1 \sin^2 \pi t dt = \int_0^1 \frac{1-\cos 2\pi t}{2} dt = \frac{t}{2}\bigg|_0^1 = \frac{1}{2}$$

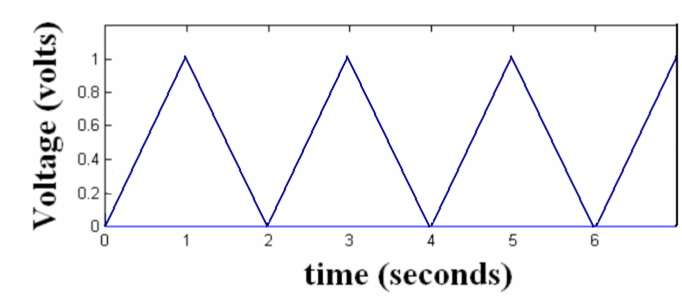
With only these terms, the approximation is:

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \pi t$$

The figure shows this approximation:



Problem 2.- Obtain the first three terms (a_o, a_1 and b_1): of the Fourier series representation of the saw-tooth signal shown in the figure



Solution: The saw-tooth signal can be described by:

$$f(t) = \begin{cases} t & if\ 0 < t < 1 \\ 2 - t & if\ 0 < t < 1 \end{cases}$$

With a period of T=2s, so the Fourier components are:

$$a_o = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{2}\left(\int_0^1 tdt + \int_1^2 (2-t)dt\right) = 1$$

$$a_1 = \frac{2}{T}\int_0^T (\cos \pi t) f(t)dt = \frac{2}{2}\left(\int_0^1 t\cos \pi t dt + \int_1^2 (2-t)\cos \pi t dt\right) = 2\int_0^1 t\cos \pi t dt$$

$$= 2\left(\frac{1}{\pi}\int_0^1 td\sin \pi t\right) = \frac{2}{\pi}\left(t\sin \pi t\Big|_0^1 - \int_0^1 \sin \pi t dt\right) = \frac{2}{\pi}\frac{\cos \pi t}{\pi}\Bigg|_0^1 = -\frac{4}{\pi^2}$$

$$b_1 = \frac{2}{T}\int_0^T (\sin \pi t) f(t)dt = \frac{2}{2}\left(\int_0^1 t \sin \pi t dt + \int_1^2 (2-t)\sin \pi t dt\right) = 0$$

With these terms, the approximation is:

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos \pi t$$

Notice how good the approximation is with only these two terms.