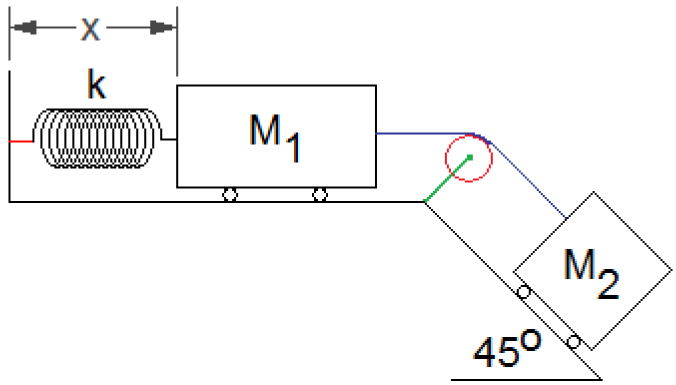
Classical Mechanics

Lagrangians

Problem 1.- Write down the Lagrangian of the following system. Assume the un-stretched length of the spring is x_o and ignore friction



Solution: The Lagrangian will be $L = \frac{1}{2}(M_1 + M_2)\dot{x}^2 - \frac{1}{2}k(x - x_\circ)^2 + mgx\sin 45^\circ$

Problem 2.- A bead is constrained to slide on a frictionless hoop of radius R which rotates about a vertical axis with constant angular velocity ω.

a) Write down the Lagrangian of this mechanical system and its equation of motion.
b) If the coordinate z is constant, calculate its value.

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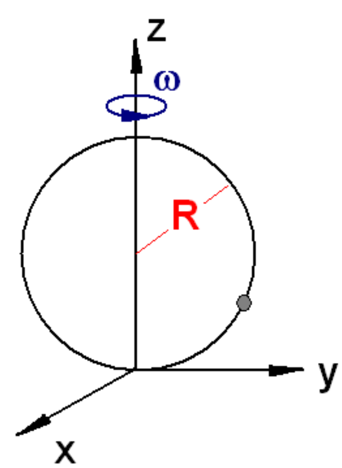
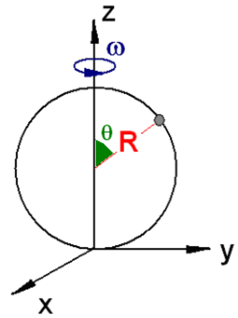


Solution: a) We can choose the angle θ to describe the position of the bead:



To find the Lagrangian we need to write down the kinetic energy and the potential energy in terms of this variable.

Potential energy: The only potential energy is gravitational. Choosing the XY plane as the reference, the P.E. is given by:

$$PE = mgz = mgR(1+\cos\theta) \quad \text{as} \quad z = R(1+\cos\theta)$$

Kinetic energy: There are two components: Kinetic energy due to the tangential velocity around the hoop and due to tangential velocity around the z-axis:

$$KE = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m(R\sin\theta)^2\omega^2$$

You can also arrive to the same equation by noticing:

$$x = R\sin\theta\cos\omega t \rightarrow \dot{x} = R\dot{\theta}\cos\theta\cos\omega t - R\omega\sin\theta\sin\omega t$$
$$y = R\sin\theta\sin\omega t \rightarrow \dot{y} = R\dot{\theta}\cos\theta\sin\omega t + R\omega\sin\theta\cos\omega t$$
$$z = R + R\cos\theta \rightarrow \dot{z} = -R\dot{\theta}\sin\theta$$

Then:

$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}mR^2\left[(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t)^2 + (\dot{\theta}\cos\theta\sin\omega t + \omega\sin\theta\cos\omega t)^2 + (-\dot{\theta}\sin\theta)^2\right]$$
$$= \frac{1}{2}mR^2\left[\dot{\theta}^2\cos^2\theta + \omega^2\sin^2\theta + \dot{\theta}^2\sin^2\theta\right] = \frac{1}{2}mR^2\left[\dot{\theta}^2 + \omega^2\sin^2\theta\right]$$

Putting all this together:

$$L = \frac{1}{2}mR^2\left[\dot{\theta}^2 + \omega^2\sin^2\theta\right] - mgR(1+\cos\theta)$$

To find the equation of motion we first take derivatives

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta} \text{, and } \frac{\partial L}{\partial \theta} = mR^2\left[\omega^2 \sin\theta\cos\theta\right] + mgR\sin\theta$$

The equation will be $mR^2\ddot{\theta} = mR^2\left[\omega^2 \sin\theta\cos\theta\right] + mgR\sin\theta$

Simplifying we get $\ddot{\theta} = \left[\omega^2 \sin\theta\cos\theta\right] + \frac{g}{R}\sin\theta$

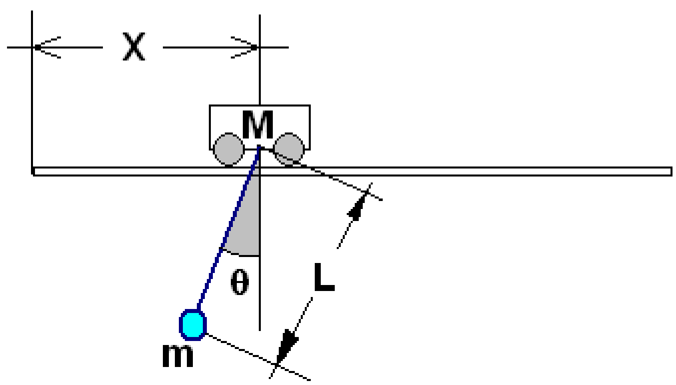
b) If the coordinate z is constant then $\ddot{\theta} = 0$, so $0 = \left[\omega^2 \sin\theta\cos\theta\right] + \frac{g}{R}\sin\theta$

One possible solution is $\sin\theta = 0$, which means that z=0 or z=2R. The other possibility is

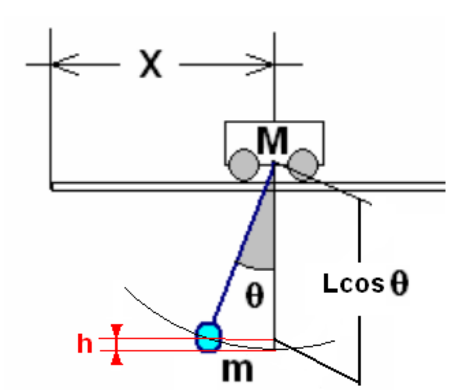
$0 = \omega^2 \cos\theta + \frac{g}{R} \rightarrow \cos\theta = -\frac{g}{\omega^2 R}$, which means that $z = R - \frac{g}{\omega^2}$

This solution will only exist if $\frac{g}{\omega^2 R} \le 1$

Problem 3.- Find the Lagrangian of the following mechanical system. Then write down the equation of motion for the variable X and interpret what it means.



Solution:



The potential energy is only due to the mass m. If we take the lowest point as the reference, the potential energy is given by:

$$P.E. = mgL(1-\cos\theta)$$

The kinetic energy of the mass M is simple, just one half of the mass times the speed squared:

$$K.E._M = \frac{1}{2}M\dot{x}^2$$

To calculate the kinetic energy of the mass m we need its coordinates:

$$x_m = x - L\sin\theta$$

$$y_m = -L\cos\theta$$

Now we take derivatives with respect to time to get the velocities:

$$\dot{x}_m = \dot{x} - L\dot{\theta}\cos\theta$$

$$\dot{y}_m = L\dot{\theta}\sin\theta$$

The kinetic energy can now be calculated:

$$K.E._m = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) =$$

$$= \frac{m}{2}\left[\dot{x}^2 - 2\dot{x}L\dot{\theta}\cos\theta + L^2\dot{\theta}^2\right]$$

Combining these results, we get the Lagrangian of the system:

$$L = \frac{m}{2}\left[\dot{x}^2 - 2\dot{x}L\dot{\theta}\cos\theta + L^2\dot{\theta}^2\right] + \frac{M\dot{x}^2}{2} - mgL(1-\cos\theta)$$

The equation of motion of X is given by:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

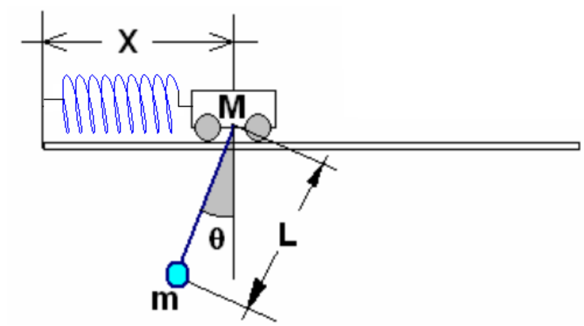
However, the first term is zero, so the second term is zero too and that means:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \rightarrow \frac{\partial L}{\partial \dot{x}} = \text{constant}$$

$$m\left(\dot{x} - L\dot{\theta}\cos\theta\right) + M\dot{x} = \text{constant}$$

You can interpret this equation as the fact that momentum in the x-direction is conserved, because the first term is the momentum of mass m and the second term is the momentum of mass M. You certainly should expect this because the only external forces acting on the system (the weights and normal force) are vertical.

Problem 3a.- Find the Lagrangian of the following mechanical system. Consider the unstretched length of the spring to be X_o.



Solution: Similar to the previous, but in the Lagrangian you need to add the potential energy due to the spring:

$$L = \frac{m}{2}\left[\dot{x}^2 - 2\dot{x}L\dot{\theta}\cos\theta + L^2\dot{\theta}^2\right] + \frac{M\dot{x}^2}{2} - mgL(1-\cos\theta) - \frac{1}{2}k(X - X_o)^2$$

Problem 4.- A particle of mass *m* on the Earth's surface is confined to move on the quartic curve $y = ax^4$ where y is up. Find the Lagrangian for the particle.

Solution: The Lagrangian can be calculated by subtracting the potential energy from the kinetic energy.

The kinetic energy is given by: $K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

The potential energy is given by: $P.E. = mgy$

So the Lagrangian can be written as: $L = K.E. - P.E. = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

However, we do not need two variables to specify the position of the particle, because the particle is restricted to move on a curve. To eliminate "x" notice that:

$$y = ax^4 \rightarrow \dot{y} = 4ax^3\dot{x} \rightarrow \dot{x} = \frac{\dot{y}}{4ax^3} \rightarrow \dot{x}^2 = \frac{\dot{y}^2}{16a^2x^6} = \frac{\dot{y}^2}{16a^{1/2}y^{3/2}}$$

With this change of variable in the Lagrangian we get:

$$L = \frac{1}{2}m\left(\frac{\dot{y}^2}{16a^{1/2}y^{3/2}} + \dot{y}^2\right) - mgy = \frac{1}{2}m\dot{y}^2\left(\frac{1}{16a^{1/2}y^{3/2}} + 1\right) - mgy$$

Problem 4a.- A particle of mass m on the Earth's surface is confined to move on the parabolic curve $y = ax^2$ where y is up. What is the Lagrangian for the particle?

Solution: The Lagrangian can be calculated by subtracting the potential energy from the kinetic energy.

The kinetic energy is: $K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

The potential energy is: $P.E. = mgy$

So the Lagrangian is: $L = K.E. - P.E. = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

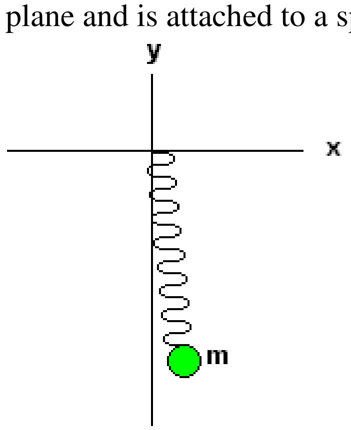
However, we do not need two variables to specify the position of the particle, because the particle is restricted to move on a curve. To eliminate "x" notice that:

$$y = ax^2 \rightarrow \dot{y} = 2ax\dot{x} \rightarrow \dot{x} = \frac{\dot{y}}{2ax} \rightarrow \dot{x}^2 = \frac{\dot{y}^2}{4a^2x^2} = \frac{\dot{y}^2}{4ay}$$

With this change of variable in the Lagrangian we get:

$$L = \frac{1}{2}m\left(\frac{\dot{y}^2}{4ay} + \dot{y}^2\right) - mgy = \frac{1}{2}m\dot{y}^2\left(\frac{1}{4ay} + 1\right) - mgy$$

Problem 5.- Write down the Lagrangian of a mass **m** that is free to move in a two dimensional plane and is attached to a spring of un-stretched length Lo and spring constant **k**.



Solution: The kinetic energy will have two components:

$$\text{K.E.} = \frac{1}{2}\text{m}\left(\dot{\text{x}}^2 + \dot{\text{y}}^2\right)$$

The potential energy should include the energy in the spring and the gravitational potential energy:

$$P.E. = \frac{1}{2}k\left[\sqrt{x^2 + y^2} - L_o\right]^2 + mgy$$

Therefore, the Lagrangian is:

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}k\left[\sqrt{x^2+y^2}-L_o\right]^2-mgy$$

Problem 6.- The Lagrangian of a mechanical system is found to be

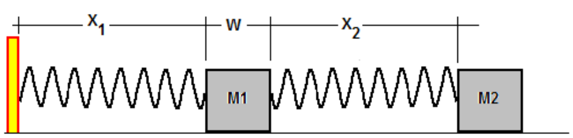
$$L=\frac{1}{2}m\dot{x}^2-mgx-\frac{1}{2}kx^2$$

Find the equation of motion of the system

Solution: $\frac{\partial L}{\partial \dot{x}}=m\dot{x}\rightarrow\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=m\ddot{x}$ and $\frac{\partial L}{\partial x}=-mg-kx$, so the equation of motion is

$m\ddot{x}=-mg-kx$ or $\ddot{x}=-g-\frac{kx}{m}$

Problem 7.- Two masses are connected together by a spring of constant k and un-stretched length Xo. The mass on the left side is also connected to a wall by an identical spring. Find the Lagrangian of the system and the equations of motion.



Solution:

$$L=\frac{1}{2}M_1\dot{x}_1{}^2+\frac{1}{2}M_2(\dot{x}_1+\dot{x}_2)^2-\frac{1}{2}k(x_1-x_0)^2-\frac{1}{2}k(x_2-x_0)^2$$

The equation of motion for x_1

$$\frac{\partial L}{\partial \dot{x}_1}=M_1\dot{x}_1+M_2(\dot{x}_1+\dot{x}_2)\rightarrow\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}=M_1\ddot{x}_1+M_2(\ddot{x}_1+\ddot{x}_2)$$

And $\frac{\partial L}{x_1}=-k(x_1-x_0)$, so the equation is

$$M_1\ddot{x}_1+M_2(\ddot{x}_1+\ddot{x}_2)=-k(x_1-x_0)$$

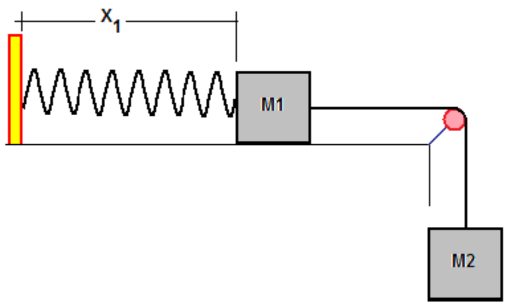
The equation of motion for x_2

$$\frac{\partial L}{\partial \dot{x}_2}=M_2(\dot{x}_1+\dot{x}_2)\rightarrow\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}=M_2(\ddot{x}_1+\ddot{x}_2)$$

And $\frac{\partial L}{x_2} = -k(x_2 - x_0)$, so the equation is

$$M_2(\ddot{x}_1 + \ddot{x}_2) = -k(x_2 - x_0)$$

Problem 8.- Write down the Lagrangian of this mechanical system and its equation of motion. Consider the un-stretched length of the spring to be X_o.



Solution: *Potential energy:* There are two sources of potential energy, gravitational and in the spring. Choosing the reference when $x_1 = 0$, the P.E. is given by:

$$PE = -M_2 g x_1 + \frac{1}{2}k(x_1 - x_o)^2$$

Kinetic energy: There are two components:

$$KE = \frac{1}{2}(M_1 + M_2)\dot{x}_1^{\,2}$$

$$L = \frac{1}{2}(M_1 + M_2)\dot{x}_1^{\,2} + M_2 g x_1 - \frac{1}{2}k(x_1 - x_o)^2$$

To find the equation of motion we first take derivatives

$$\frac{\partial L}{\partial \dot{x}_1} = (M_1 + M_2)\dot{x}_1 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = (M_1 + M_2)\ddot{x}_1 \text{, and } \frac{\partial L}{\partial x_1} = M_2 g - k(x_1 - x_o)$$

Then the equation will be

$$(M_1 + M_2)\ddot{x}_1 = M_2 g - k(x_1 - x_o)$$

Problem 9.- Find the equation of motion of a system whose Lagrangian is given by:

$$\mathrm{L} = \frac{1}{2} I\dot{\theta}^2 - \frac{1}{2} K(\theta - \theta_o)^2$$

Solution: There is only one variable in this Lagrangian so there will be only one equation of motion:

$$\frac{\partial \mathrm{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -K(\theta - \theta_o) - \frac{d}{dt}\left(I\dot{\theta}\right) \rightarrow \qquad I\ddot{\theta} = -K(\theta - \theta_o)$$

Problem 10.- Find the Lagrangian of the following mechanical system. Consider the un-stretched length of the spring to be X_o. Ignore friction.