Classical Mechanics

Perturbation theory

Problem 1.- Using first order perturbation theory, find an approximate solution to the oscillation of a particle in the potential:

$$U = \frac{1}{2}kx^2 - \frac{1}{5}\lambda m x^5$$

Solution: Given the potential U, the force is given by: $F = -\frac{dU}{dx} = -kx + \lambda m x^4$, which gives the equation of motion: $m\frac{d^2x}{dt^2} = -kx + \lambda m x^4$, or equivalently: $\frac{d^2x}{dt^2} = -\omega_o^2 x + \lambda x^4$, where $\omega_o = \sqrt{\frac{k}{m}}$

Using first order perturbation theory, we assume that the solution can be written as: $x = x_o + \lambda x_1$, and putting this in the equation of motion:

$$\ddot{x}_o + \lambda \ddot{x}_1 = -\omega_o^2 x_o - \omega_o^2 \lambda x_1 + \lambda (x_o{}^4 + 4\lambda x_o{}^3 x_1 + 6\lambda^2 x_o{}^2 x_1{}^2 + 4\lambda^3 x_o x_1{}^3 + \lambda^4 x_1{}^4)$$

In first order we can ignore λ^2 or higher powers, so we get two equations: one for the terms independent of λ and one for terms linear in λ:

$$\ddot{x}_o = -\omega_o^2 x_o$$

$$\ddot{x}_1 = -\omega_o^2 x_1 + x_o{}^4$$

The solution to the first equation is the usual: $x_o = A\cos\omega_o t$ where we omitted the phase shift for convenience. To solve the second equation, we insert this solution to get:

$$\ddot{x}_1 = -\omega_o^2 x_1 + (A\cos\omega_o t)^4 = -\omega_o^2 x_1 + A^4 \cos^4 \omega_o t$$

Here we use the identity $\cos^4 \omega_o t = \frac{3 + 4\cos 2\omega_o t + \cos 4\omega_o t}{8}$, so

$$\ddot{x}_1 = -\omega_o^2 x_1 + A^4 \frac{3 + 4\cos 2\omega_o t + \cos 4\omega_o t}{8}$$

We can try the solution: $x_1 = B + C\cos 2\omega_o t + D\cos 4\omega_o t$, which gives:

$$-4\omega_o^2 C\cos 2\omega_o t - D16\omega_o^2 \cos 4\omega_o t + \omega_o^2 (B + C\cos 2\omega_o t + D\cos 4\omega_o t) = A^4 \frac{3 + 4\cos 2\omega_o t + \cos 4\omega_o t}{8}$$

This gives us the values of B, C and D:

$$B\omega_o^2 = A^4 \frac{3}{8} \rightarrow B = \frac{3A^4}{8\omega_o^2} \qquad -4\omega_o^2 C + \omega_o^2 C = A^4 \frac{4}{8} \rightarrow C = -\frac{A^4}{6\omega_o^2}$$

$$-D16\omega_o^2 + \omega_o^2 D = A^4 \frac{1}{8} \rightarrow D = -\frac{A^4}{120\omega_o^2}$$

The solution then becomes:

$$x = A\cos\omega_o t + \lambda \frac{A^4}{\omega_o^2}\left(\frac{3}{8} - \frac{\cos 2\omega_o t}{6} - \frac{\cos 4\omega_o t}{120}\right)$$